Crystals and Groups

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Outline

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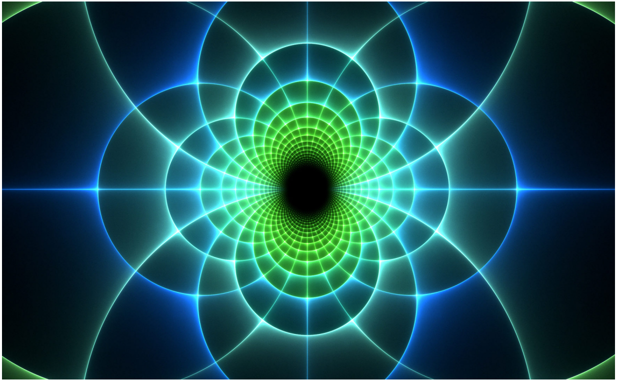
1. What is symmetry?
2. 2D: Tilings and wallpaper groups
3. 3D: Polyhedra and crystallographic groups
4. Unexpected symmetries: Penrose tilings and quasi-crystals

What is symmetry?

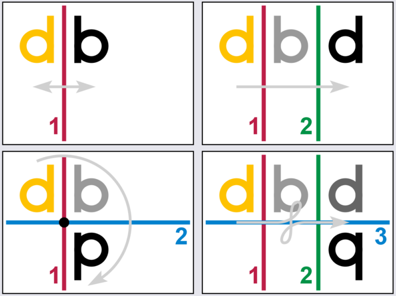


1. How do we define symmetry?
2. How do we detect symmetry?
3. How do we classify symmetry?

Symmetry as transformation

An isometry is a transformation that preserves the distance between points.

Example: isometries of the Euclidean plane

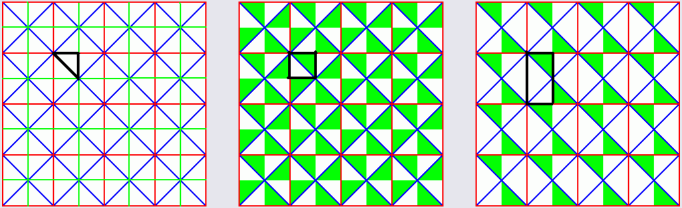


The set of isometries of a metric space X *has the structure of a group*. It is called the symmetry group of X, and denoted by $\mathrm{Isom}(\mathrm{X})$.

Symmetry as a geometric property

Symmetry appears as *repetition of a pattern*. A good way to understand symmetry: detect the fundamental domain.

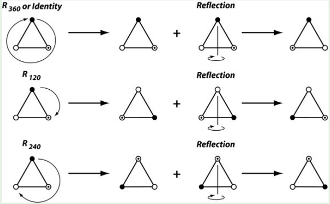
Fundamental Domains



The group of symmetry of a repeated pattern in a metric space X is the subgroup of $\mathrm{Isom}(X)$ that identifies the fundamental domains of that pattern.

Examples

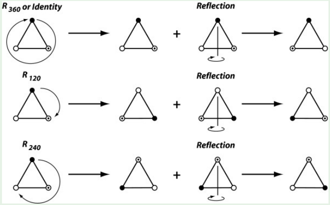
Example (Triangle)



$$\mathrm{Isom}(X) = D_3.$$

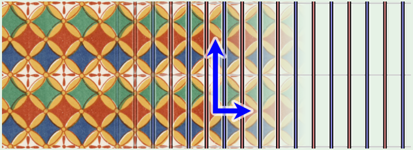
Examples

Example (Triangle)



$$\mathrm{Isom}(X) = D_3.$$

Example (Wallpaper)



$$\mathbf{Z} \times D_\infty \subset \mathrm{Isom}(\mathbf{R}^2).$$

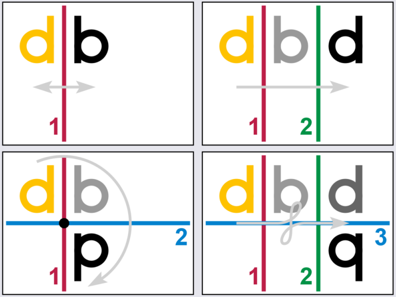
Where are we?

Symmetries in 2D

Recall

Let X be the Euclidean plane $\mathbf{R}^2$. The group $\mathrm{Isom}(\mathbf{R}^2)$ is generated by:

- translations
- rotations
- reflections



The Alhambra wallpapers

The Alhambra (mid-13th century)

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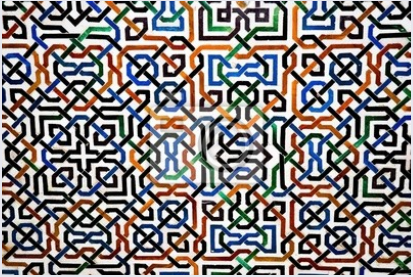
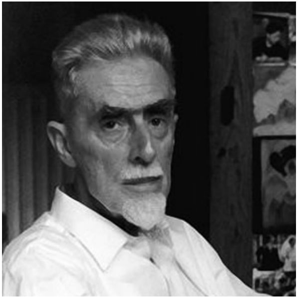
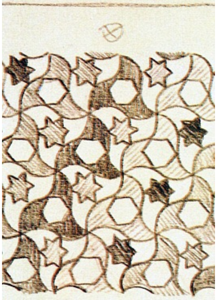


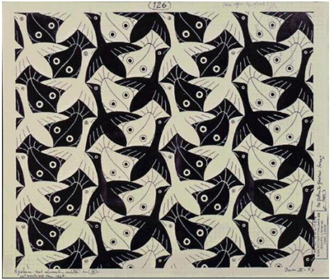
Maurits Cornelis Escher (1898 - 1972)





'It remains an extremely absorbing activity, a real mania to which I have become addicted, and from which I sometimes find it hard to tear myself away.' M.C.Escher

Escher wallpapers

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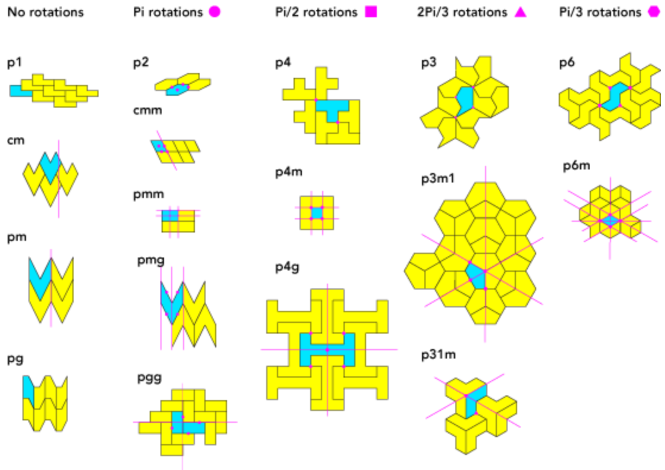
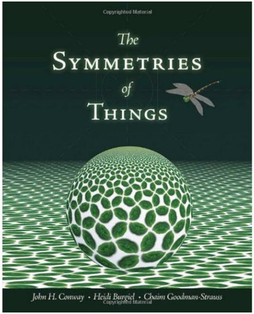
Types of symmetries

Let X be the Euclidean plane $\mathbf{R}^2$. Let P be a pattern in X invariant by two linearly independent translations.
Then the group $\mathrm{Isom}(P) \subset \mathrm{Isom}(\mathbf{R}^2)$ has one of the following 17 types:



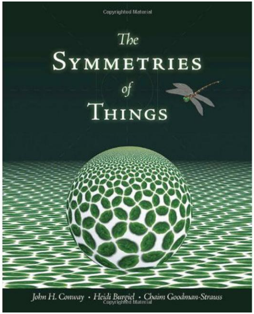
Conway's magic theorem



The Magic Theorem

Symbol	Cost ($)	Symbol	Cost ($)
○	2	$*$ or ×	1
2	$\frac{1}{2}$	2	$\frac{1}{4}$
3	$\frac{2}{3}$	3	$\frac{1}{3}$
4	$\frac{3}{4}$	4	$\frac{3}{8}$
5	$\frac{4}{5}$	5	$\frac{2}{5}$
6	$\frac{5}{6}$	6	$\frac{5}{12}$
⋮	⋮	⋮	⋮
N	$\frac{N-1}{N}$	N	$\frac{N-1}{2N}$
∞	1	∞	$\frac{1}{2}$

Table 3.1. Costs of symbols in signatures.



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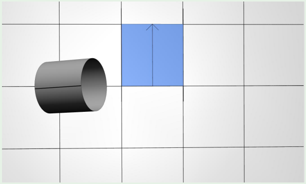
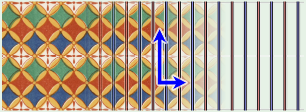
Example ($**$ orbifold $\leftrightarrow$ *pm* wallpaper group)



$\leftrightarrow$

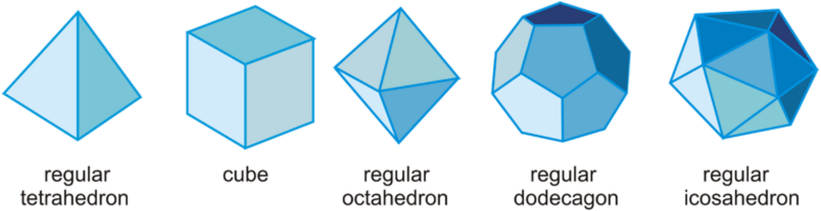


Where are we?

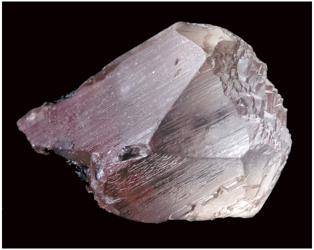
Platonic solids

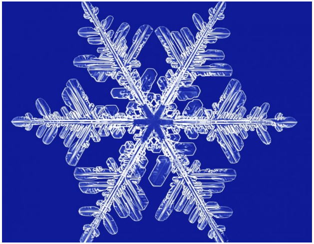
Exercise

What are the symmetry groups of the following objects?



Crystals

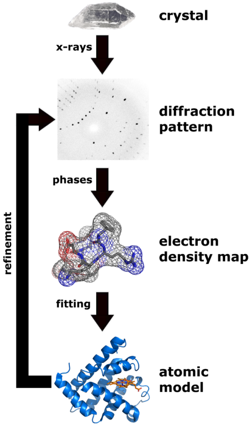




Crystals

Definition

A crystal is a three-dimensional periodic arrangement of atoms with translational periodicity along its three principal axes.



To understand the atomic structure of crystals, one has to figure out the symmetries of its diffraction pattern.

This pattern is invariant by three linearly independent translations in $\mathbf{R}^3$.

Question

How many symmetry groups (=crystallographic groups) for these patterns?

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Answer

230

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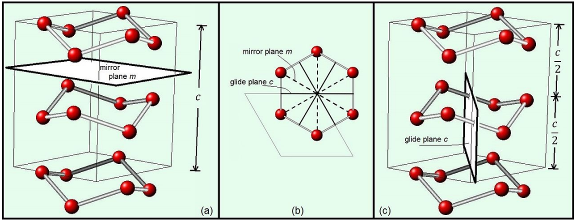
Question

How many symmetry groups (=crystallographic groups) for these patterns?

Answer

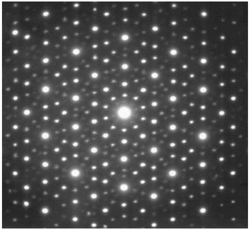
230



Atomic model of ice crystals.

Where are we?

A Nobel prize discovery



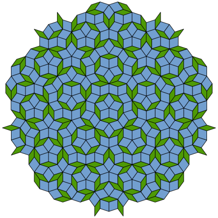
In 1982 Dan Shechtman detects crystals with icosahedral symmetries, "forbidden" by the classical theory.

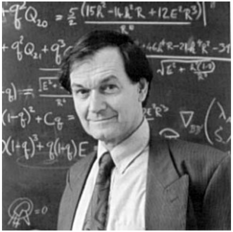
Shechtman's findings were received with skepticism, but slowly accepted.

In 2011 he was awarded the Nobel Prize in Chemistry, for the discovery of *quasicrystals*.

Penrose tilings

1970s: Roger Penrose discovered *quasiperiodic tiles*.





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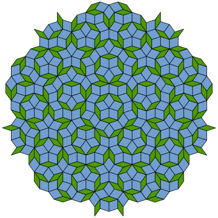

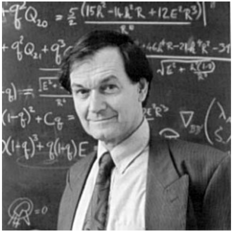

'...as soon as Steinhardt showed me the diffraction patterns that Shechtman had found I was happy to believe that nature had found a way around the problem.' R. Penrose