A Berkovich-analytic approach to models of curves over DVRs

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Plan

1. Berkovich curves and models
2. Semi-stable reduction
3. Regular models and Saito's criterion
4. The potentially multiplicative case

Berkovich analytification

Let $(K, |\cdot|_K)$ be a complete non-archimedean field, $R = \{x \in K : |x|_K \leq 1\}$, $\mathfrak{m} = \{x \in K : |x|_K < 1\}$, and $k = R/\mathfrak{m}$.

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Examples

- $K = \mathbb{Q}_p, R = \mathbb{Z}_p, k = \mathbb{F}_p$
- $K = \mathbb{C}_p, R = \mathcal{O}_{\mathbb{C}_p}, k = \overline{\mathbb{F}}_p$
- k any field, $K = k((t)), R = k[[t]]$
- K any field, with the trivial valuation, $R = K$, $k = K$.

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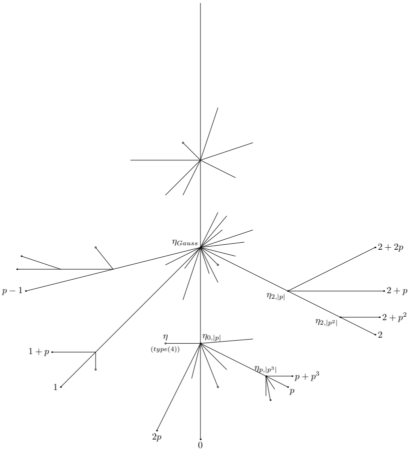
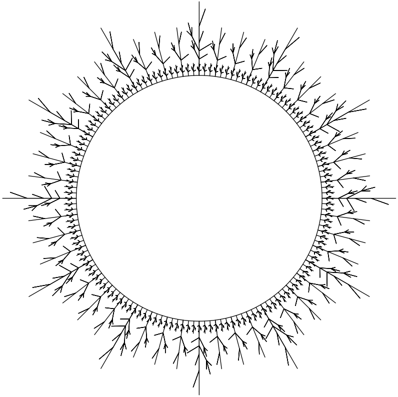
Let X be a variety over K. Its Berkovich analytification X^{an} is the set of pairs $(x, |\cdot|)$ with $x \in X$ and $|\cdot| : \kappa(x) \to \mathbb{R}_{\geq 0}$ absolute value extending $|\cdot|_K$.

Remark

- *X^{an} can be endowed with the structure of a locally ringed space (topology + structure sheaf),*
- *$X \to X^{\mathrm{an}}$ can be made into a functor*



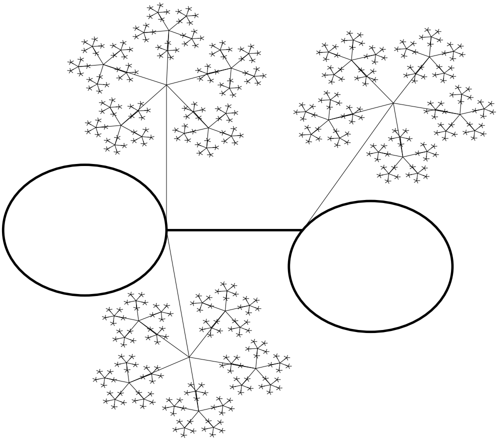
$\mathbb{P}^{1,\mathrm{an}}_{\mathbb{Q}_p}$



E^{an}

Global structure

Let C be a smooth projective curve over K. The analytification C^{an} retracts on a finite graph, called the skeleton of C.



Berkovich curves

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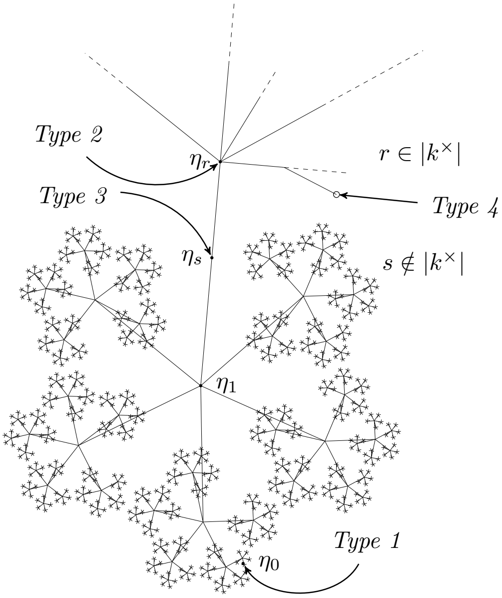
Local structure

There are 4 types of points in a Berkovich curve:

- Type 1: $(x, |\cdot|_{\kappa(x)})$ with $x \in C$ closed point.
- Type 2: the points $y \in C^{\mathrm{an}}$ with a neighborhood V such that $V \setminus y$ has an infinite number of connected components
- Type 3: the points $y \in C^{\mathrm{an}}$ with a neighborhood V such that $V \setminus y$ has two connected components
- Type 4: the points $y \in C^{\mathrm{an}}$ with a neighborhood V such that $V \setminus y$ is connected and y is not of type 1.

Types of points



Models

Let K be a discretely valued field. Let C be a smooth projective curve over K.

Definition

A model *of C is a flat, proper curve $\mathcal{C}$ over R such that $\mathcal{C} \times_R K \cong C$. The k-scheme $\mathcal{C}_k := \mathcal{C} \times_R k$ is called the* special fiber *of $\mathcal{C}$, while C is its* generic fiber.

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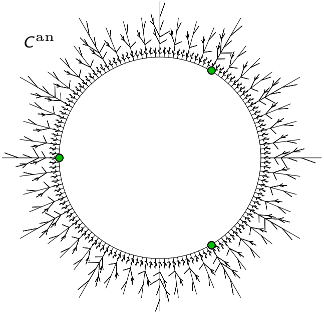
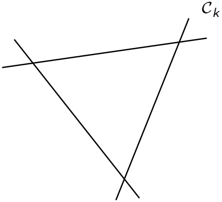
Proposition

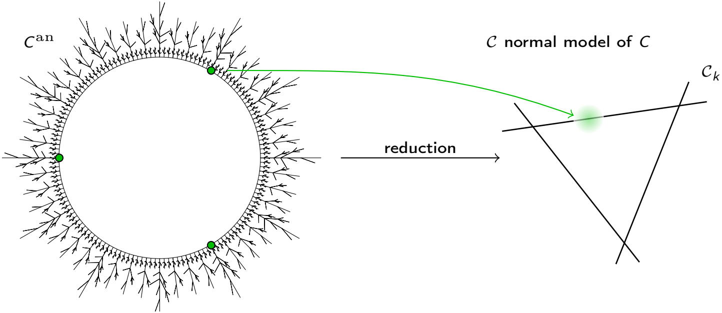
There is an order preserving bijection:

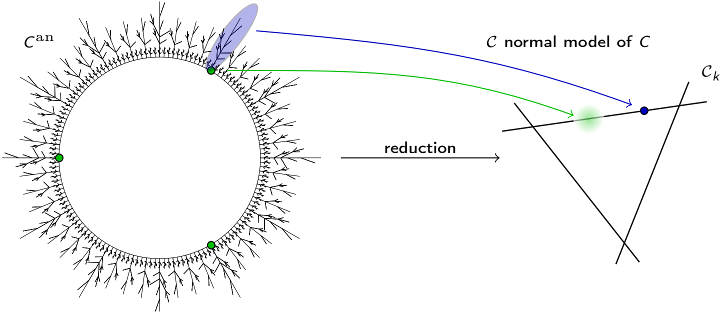
$$\{\text{normal models of } C\} \quad \longleftrightarrow \quad \{\text{non-empty finite subsets of } C^{\mathrm{an}} \text{ containing only type 2 points}\}$$



$\mathcal{C}$ **normal model of** C




C^{an}
$\mathcal{C}$ normal model of C
$\mathcal{C}_k$
reduction



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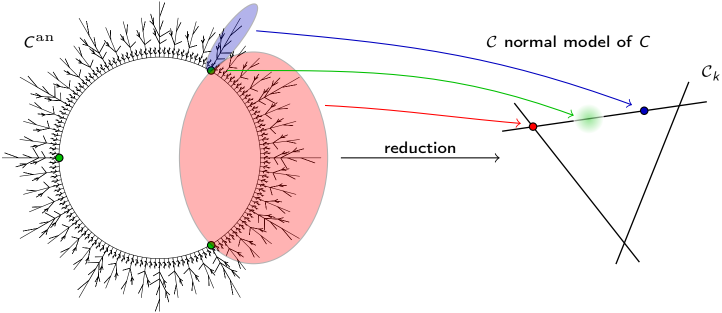
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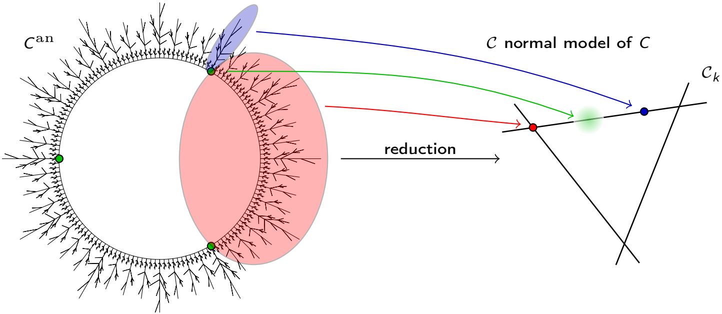

Let $x \in S$ be a type 2 point. The multiplicity $m(x)$ of the irreducible component $\mathcal{C}_{k,x}$ corresponding to x does not depend on the choice of a model!

The semi-stable reduction theorem

Let C be a smooth projective curve over K. A semi-stable model of C is a model $\mathcal{C}$ whose special fiber $\mathcal{C}_k := \mathcal{C} \times_R k$ satisfies:

- $\mathcal{C}_k$ is reduced
- $\mathcal{C}_k$ has at most double nodal singularities.

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Question

What is the minimal extension $L|K$ such that C_L has a semi-stable model?

Minimal triangulations

Definition

A triangulation of C^{an} is a finite set $S \subset C^{\mathrm{an}}$ such that $C^{\mathrm{an}} \setminus S$ is a union of virtual discs and finitely many virtual annuli.

Proposition

If the genus $g(C)$ is at least two, then C^{an} has a minimal triangulation S_{min-tr}.

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Theorem (Fantini – T., 2020)

1. *The minimal extension $L|K$ yielding semi-stability is the minimal extension "resolving the multiplicities" at all points of S_{min-tr}*
2. $d := \mathrm{lcm}\{m(x) : x \in S_{min-tr}\} | [L : K]$
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Elements of proof

- Behaviour of S_{min-tr} after base-change
- Explicit descriptions of tame forms of discs and annuli (Ducros '13, Fantini - T. '18)

Regular models

Resolution of singularities for surfaces $\implies$ There exists a regular model of C (no need to base-change!).

In fact, there is a minimal regular model with strict normal crossings $\mathcal{C}_{min-snc}$ (which induces a set of type 2 points $S_{min-snc} \subset C^{\mathrm{an}}$).

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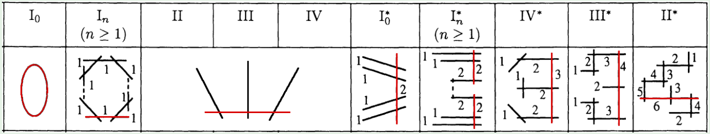
Definition

An irreducible component of a model $\mathcal{C}$ is called principal if it is of genus > 0 or if it intersects the rest of $\mathcal{C}$ in at least three points. The quantity

$$e(C) = \mathrm{lcm}\{m(E) : \text{E is a principal component of } \mathcal{C}_{min-snc}\}$$

is the stabilization index of C.

Example



The tame case

Theorem (Fantini - T., 2020)

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As a corollary, we get a different proof of:

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The minimal extension $L|K$ yielding semi-stability is tamely ramified if and only if $(e(C), p) = 1$.

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Question (wide open)

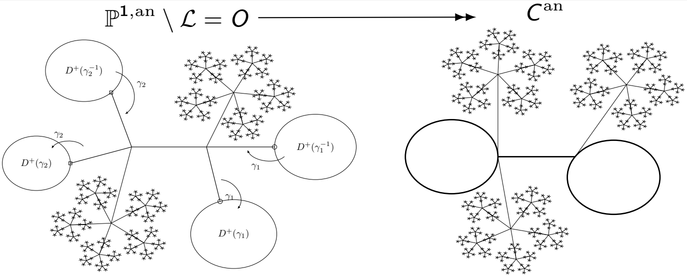
What happens when $L|K$ is wild?

Mumford curves

A curve C over K is called Mumford curve if it has a semi-stable model $\mathcal{C}$ such that the irreducible components of $\mathcal{C}_k$ are projective lines.

Theorem (Mumford 1972)

C is a Mumford curve $\iff$ there exists an open dense subset $O \subset \mathbb{P}^{1,\mathrm{an}}_K$ and a free group $\Gamma \subset PGL_2(K)$ with $\Gamma \backslash O \cong C^{\mathrm{an}}$.



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Theorem (Berkovich 1990)

C is a Mumford curve of genus g $\iff$ C^{an} admits a continuous retraction on a graph of Betti number g.

Stabilization index for potentially Mumford curves

Question (Halle–Nicaise)

Let C be a form of a Mumford curve, of index one and let $L|K$ be the minimal extension yielding semi-stable reduction. Do we have $[L : K] = e(C)$?

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Theorem (Obus–T., 2021)

1. *There exist C as above such that $[L : K] \neq e(C)$*
2. *Let C be a form of a Mumford curve and let $L|K$ be the minimal extension yielding semi-stable reduction. Then $e(C) \mid [L : K]$.*

Elements of proof.

- Uniformization of C_L^{an}
- Action of $\mathrm{Gal}(L|K)$ over C_L^{an} (global and local!)
- Resolution of quotient singularities in the weak wild case (Obus-Wewers).

□

That's all (for now)

Thank you!