Berkovich analytification and tropicalization of algebraic varieties

Daniele Turchetti

University of Warwick

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Motivation

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Studies geometric objects *locally defined by polynomials*.

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- Do the cohomology groups $H^{4g-6}(M_g(\mathbb{C}), \mathbb{Q})$ vanish for almost all g's?
- Is a general quartic fivefold $X \hookrightarrow \mathbb{P}^6(\mathbb{C})$ rational or irrational?
- Can we find explicit uniform bounds for the number of points of a curve of genus $g > 1$ over a number field?

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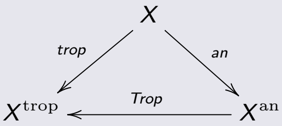
Tropicalization and analytification

To an algebraic variety X, one can associate a tropicalization X^{trop} and an analytification X^{an} fitting in a commutative diagram:



Non-archimedean fields

A valued field $(K, ||\cdot||)$ is a field together with a function

$$||\cdot|| : K \to \mathbb{R}_{\geq 0}$$

satisfying

- $||x|| = 0$ iff $x = 0$
- $||xy|| = ||x|| \cdot ||y||$
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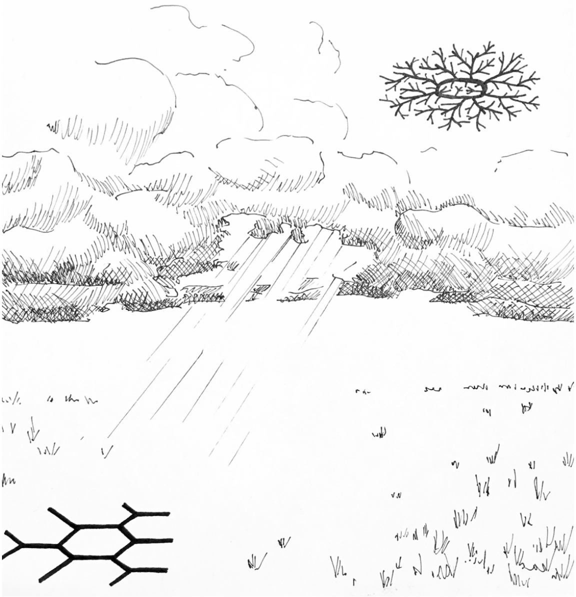
- $(\mathbb{C}(t), |\cdot|_t)$ –> good for studying families of complex varieties
- $(\mathbb{Q}, |\cdot|_p)$ –> good for arithmetic geometry
- $(k, |\cdot|_0)$ –> good for dynamics, birational geometry, and to study singularities

Non-archimedean analytification and tropicalization: a metaphor



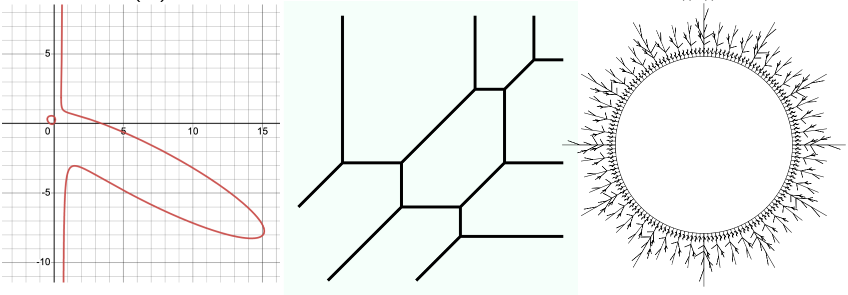
"Berkovich analytic spaces are heavenly abstract objects which can be viewed by earthly beings through their tropical shadows."

From M. Brandt's thesis (supervised by B. Sturmfels)

Example (elliptic curves)

Let C be defined by: $2y^2 + x + xy - y - \pi xy^2 - \pi x^2 y + \pi x^2 - x^3 = 0$.

$C(\mathbb{R})$ $\quad C^{\mathrm{trop}}_{\mathbb{Q}(\pi)}$ $\quad C^{\mathrm{an}}_{\mathbb{Q}((\pi))}$



A brief history of non-archimedean geometry

- 1960's Tate introduces *rigid analytic geometry*
- 1970's Raynaud links rigid spaces and formal algebraic geometry
- ∼1990 Berkovich conceives a new theory using spaces of valuations and spectral theory
- 1990's Huber's *adic spaces* generalize Berkovich's theory
- ∼2010 Poineau develops the theory of Berkovich spaces over $\mathbb{Z}$

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Applications to

- Arithmetic geometry: local Langlands program (étale cohomology on Berkovich spaces) and *p*-adic Hodge theory (Scholze's perfectoid spaces)
- Combinatorial algebraic geometry (via connections to toric and tropical geometries)
- String theory (degeneration of Calabi–Yau, mirror symmetry, SYZ fibration)
- Dynamical systems and potential theory (dynamics on Berkovich spaces)
- *p*-adic differential equations (radii of convergence on Berkovich curves)
- Inverse Galois problem

A brief history of tropical geometry

- 1984 Bieri, Groves, and Strebel associate polyhedral structures to ideals of $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$
- 1998 the adjective "tropical" is found for the first time in the literature (cf. *Tropical semirings* by Jean-Eric Pin)
- 2005 Mikhalkin establishes that Gromov-Witten invariants of $\mathbb{P}^2(\mathbb{C})$ are controlled by their tropical counterparts
- 2007 Tevelev defines tropical compactifications for subvarieties of the torus (later generalized to subvarieties of toric varieties)
- 2010's Tropicalization techniques are applied fruitfully to moduli spaces
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Tropical approach applies also to

- Geometric group theory (Bieri–Groves)
- Biology (e.g phylogenetic trees)
- Algebraic statistics
- Economics
- Training neural networks

The "Berkovich + tropical" approach

- 1990's Berkovich shows that a "nice" Berkovich space retracts continuously on a finite polyhedral complex (its skeleton)
- 2009 Payne shows that "Berkovich analytification is the limit of all tropicalizations"
- 2013 Tropical perspective on skeletons of Berkovich curves (Baker–Payne–Rabinoff)
- 2016 Definition of essential skeletons for varieties over $\mathbb{C}((t))$ (Nicaise–Xu)
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- Tropical obstruction to specialization of (stable) rationality (Nicaise–Ottem)
- Ramification of covers of curves (Temkin, Amini–Baker–Brugallé–Rabinoff)
- Bogomolov conjecture for totally degenerate abelian varieties (Gubler)
- Brill–Noether theory (via divisors on tropical curves)
- Cohomology of $\mathcal{M}_{g,n}$ (Chan–Galatius–Payne)
- Uniform bounds for the number of points of curves over number fields (Katz–Rabinoff–Zureick-Brown)

Plan of lectures

1. Introduction and motivation [you are here]
2. Berkovich spectra and analytification
3. Models of Berkovich spaces (or: how to study degenerations using Berkovich spaces and viceversa)
4. Curves I
5. Curves II
6. Higher dimensional varieties (or: why can't everything be as nice as in dimension 1)
7. Tropical geometry and tropicalization
8. Tropicalization and analytification
9. Berkovich + tropical moduli I ($M_{0,n}^{\mathrm{an}}$ and $M_{0,n}^{\mathrm{trop}}$?)
10. Berkovich + tropical moduli II ($M_{g,n}^{\mathrm{an}}$ and $M_{g,n}^{\mathrm{trop}}$?)

Essential Bibliography

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