Hurwitz graphs and Berkovich curves

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Abstract

Hurwitz trees were introduced by Henrio and Brewis-Wewers to understand actions of finite order over p-adic discs. We give a geometric interpretation of these objects in terms of non-archimedean analytic geometry in the sense of Berkovich. We extend the definition to the global context, introducing the notion of a Berkovich-Hurwitz graph. This allows to study liftings of branched Galois covers of (possibly singular) curves. In the end we discuss the possibility to study deformations of covers from an analytic point of view to approach lifting problems with methods coming from tropical geometry.

Résumé

Les arbres de Hurwitz ont été introduits par Henrio et Brewis-Wewers pour éclaircir les propriétés des actions d'un groupe fini sur le disque p-adique. Nous donnons ici une interprétation géométrique de ces objets dans le contexte de la géométrie de Berkovich. Ensuite, on définit les graphes de Hurwitz, qui permettent d'étudier les relèvement des revêtements ramifiés de courbes (pas forcement lisses). Avec ceci, on peut étudier les déformations de ces revêtements d'un point de vue analytique et éclaircir certains aspects de la relation entre géométrie tropicale et problèmes de relèvement.

Contents

0 Introduction

Tamely ramified local actions of finite groups on curves are completely determined by Kummer theory. The wildly ramified case is much more difficult and its complexity is effectively encoded via the theory of Hurwitz trees, as introduced in [Hen] and [BW09]. In this paper we give a non-archimedean analytic interpretation of Hurwitz trees, in the framework of Berkovich spaces. This allows to describe the features of Hurwitz trees in a homogeneous way and to describe local actions in a purely analytic fashion. Using formal patching, we are able to investigate the global situation, where finite Galois covers of curves are considered. As a result, one can apply the theory of the different function and compatible skeletons of finite morphisms of Berkovich curves to the algebraic case.

1 The local picture: Berkovich-Hurwitz trees

Let K be a complete discretely valued field of mixed characteristic $(0, p)$, let R be its valuation ring, k its residue field, that we suppose algebraically closed, and π an uniformizer generating the maximal ideal $\mathfrak{m}$. In this section we recall the definition of a Hurwitz tree associated with a local action in characteristic zero and show a description of the theory of Hurwitz trees in the framework of Berkovich non-archimedean analytic geometry.

1.1 The local lifting problem

Definition 1.1. A local action in characteristic p is a pair $(k[[t]], G)$ where $G \subset \mathrm{Aut}_k k[[t]]$ is a finite subgroup of k-automorphisms. A local action in characteristic zero is a pair $(R[[T]], G')$ where $G' \subset \mathrm{Aut}_R R[[T]]$ is a finite subgroup of continuous R-automorphisms.

We assume for simplicity that the fixed points in the rigid generic fiber of a local action in characteristic zero belong to K. This can always be done after replacing K with a finite extension. A natural way to construct local actions is to consider the action of a group G on a smooth projective curve X. In this setting, for every closed point x of X, the stabilizer of G at x acts on the local ring $\widehat{\mathcal{O}}_{X,x}$, which is a ring of formal power series.

Remark 1.2. By the structure theorem of complete discrete valuation rings in equicharacteristic, one deduce that any local action $(k[[t]], G)$ yields $k[[t]]^G \cong k[[z]]$. For a local action $(R[[T]], G')$, a result of Raynaud ([Ray99], Proposition 2.3.1) ensures that $R[[T]]^{G'} \cong R[[Z]]$. Hence, in every characteristic it is equivalent to study local actions and finite Galois covers of algebras of formal power series.

Example 1.3 (Deformation from Kummer to Artin-Schreier).
Let $G = \mathbb{Z}/p\mathbb{Z}$ act on $R[[T]]$ in such a way that the m ramification points have the same valuation. Then, the equation of the covering is of the special Kummer type $X^p = (1 + T^m)$. After performing the substitution $X = 1 + \beta U$ and $S = \beta^{\frac{p}{m}} T$, the Kummer equation becomes

$$(1+\beta U)^p = (1+\beta^p T^m) \iff (\beta U)^p + p(\beta U)^{p-1} + \cdots + p\beta U = \beta^p T^m \iff U^p + \cdots + \frac{p}{\beta^{p-1}} U = T^m.$$

When $v_p(\beta) = \frac{1}{p-1}$ the reduction mod. p of this equation is $u^p - cu = t^m$, with $c \neq 0$ namely an Artin-Schreier equation.

The deformation from Kummer to Artin-Schreier equations ([OSS89]) is used by Green and Matignon to show that every action of $\mathbb{Z}/p\mathbb{Z}$ lifts to characteristic zero [GM99, II, Theorem 4.1]. Indeed, given an equation for a covering in characteristic p one can explicitly find a Kummer equation giving rise to a lifting. The existence of deformations of a given cover are equivalent to the existence of some differential forms. This is the idea that gives rise to the definition of *Hurwitz tree* as introduced by Henrio in [Hen]. In the first part of this paper, we work with the following generalization, due to Brewis and Wewers [BW09].

In order to state the definition of a Hurwitz tree, we shall briefly establish some terminology. A *rooted tree* is a finite, oriented tree that has a unique minimal vertex (called the *root*) for the partial ordering '$\prec$' that defines the orientation. The maximal vertices, that can be multiple, are called the *leaves* of the rooted tree. Given e an edge of a rooted tree, we denote the endpoints of e by $s(e), t(e)$, in such a way that $s(e) \prec t(e)$.
A K-valued *virtual character* of a finite group G is a function $\chi : G \longrightarrow K$ that is invariant under conjugation by elements of G, sometimes also called class function. A virtual character is a (representation theoretical) *character* if it arises as the trace of a suitable finite dimensional representation of G over K.

Example 1.4. Let G be any finite group. The class function $u_G : G \longrightarrow \mathbb{Z}$ defined by

$$u_G(\sigma) = \begin{cases} -1 & \text{if } \sigma \neq 1 \\ |G| - 1 & \text{if } \sigma = 1. \end{cases}$$

gives rise to a character, called *augmentation character* of G. Indeed it is not difficult to show that it is associated with the augmentation representation.

Example 1.5. Let $G = \mathbb{Z}/p^m\mathbb{Z}$ and fix a generator σ of G. Brewis and Wewers prove that the class function $\delta_G^{mult} : G \longrightarrow \mathbb{Q}$ defined by

$$\delta_G^{mult}(\sigma^a) = \begin{cases} -\frac{p^{i+1}}{p-1} & \text{if } \mathrm{ord}_p(a) = i < m \\ 0 & \text{if } \sigma^a = 1. \end{cases}$$

gives rise to a $\mathbb{Q}$-*valued character*[1], a generalization of the notion of a character, discussed in [BW09, §2.1].

Definition 1.6 (Hurwitz tree)**.** Let G be a finite group and let K be a discrete valuation ring of mixed characteristic $(0, p)$. A G-Hurwitz tree over a field K is the datum $(T, [G_v], a_e, \delta_v)$ of:

- a metrized rooted tree T, with set of edges E, set of vertices V and root $v_0 \in V$;
- for every vertex $v \in V$, the conjugacy class $[G_v]$ of a subgroup $G_v \subset G$;
- for every edge $e \in E$, a character $a_e : G \to K$;
- for every vertex $v \in V$, a $\mathbb{Q}$-valued character $\delta_v : G \to K$

satisfying the following conditions:

HT1. one has $G_{v_0} = G$ and G_b cyclic for every leaf b. Moreover, for every $v, w \in V$ such that $v \prec w$ one has, up to conjugation, $G_w \subseteq G_v$;

HT2. for every edge $e \in E$, one has

$$a_e = \sum_{b \in B_e} u_{G_b},$$

where B_e is the set of leaves that are greater or equal than $t(e)$;

[1]To fix: this notion is not central in the theory of Hurwitz tree and we shall probably avoid it even if it is part of the original definition.

HT3. for every edge $e \in E$ one has

$$\delta_{t(e)} = \delta_{s(e)} + \epsilon_e \cdot (a_e - u_{G_{t(e)}})$$

where ϵ_e is the length of the edge e;

HT4. for every leaf $b \in B$ one has $\delta_b = \delta_{P_b}^{mult}$, where P_b is the Sylow p-subgroup of G_b.

Brewis and Wewers associate a Hurwitz tree to any local action in characteristic zero. This allows them to study a necessary condition for lifting that requires the existence of Hurwitz trees with special properties. They call it the *Hurwitz tree obstruction* to lifting.[2] The first goal of this paper is to give an alternative construction of the Hurwitz tree associated with a local action in characteristic zero, that relies on the theory of compatible skeletons for morphisms of Berkovich curves.

1.2 The Berkovich-Hurwitz tree of a local action

Let $\Lambda = (R[[T]], G)$ be a local action in characteristic zero. The inclusion $R[[T]]^G \subset R[[T]]$ can be regarded as a G-Galois cover of formal schemes $\mathrm{Spf}(R[[T]]) \to \mathrm{Spf}(R[[T]]^G)$. To this cover, we can apply a suitable *generic fiber* functor to get a G-Galois cover $\eta_\Lambda : X' \to X$ of Berkovich curves, as described in what follows. In the paper [Ber96b], Berthelot associates with every formal schemes locally formally of finite type $\mathcal{X}$ its *generic fiber*, a rigid analytic space over K. This construction, which is functorial in $\mathcal{X}$, can be adapted to get a Berkovich space, that we denote $\mathcal{X}^\beth$. This is well illustrated in [Ber96a, §1].[3] The easiest example of generic fiber of a formal scheme formally of finite type is the Berkovich open disc[4], $D^\circ \cong \mathrm{Spf}(R[[T]])^\beth$.

1.2.1 Skeletons and the metric realization

Thanks to Remark 1.2, we know that $R[[T]]^G \cong R[[Z]]$ so that the generic fiber of the cover induced by Λ is a G-Galois cover of the form $p_\Lambda : D^\circ \to D^\circ$. The fundamental result that allows us to recover the Hurwitz tree in this context is obtained by applying the theory of *skeletons* to the cover p_Λ. Most of the results in the this section are well-known, but are scattered in the literature with slightly varying assumptions. For the sake of completeness and the convenience of the reader, we recall them pointing to proper references when necessary. Recall from [Duc14, §3.1.11] that a K-analytic space X is said to be *quasi-smooth* in $x \in X$ if $\dim_{\mathcal{H}(x)} \Omega_X^1 \otimes \mathcal{H}(x) = \dim_x(X)$. It is called *quasi-smooth* if it is quasi-smooth in every point.

Definition 1.7. Let X be a quasi-smooth K-analytic curve. A *skeleton* of X is a subgraph $\Gamma_X \subset X$ such that $X \setminus \Gamma_X$ is a disjoint union of virtual open discs.
Let $\phi : X \to Y$ be a finite morphism of quasi-smooth Berkovich curves. A *skeleton* of f is a pair (Γ_X, Γ_Y) such that:

- Γ_X is a skeleton of X and Γ_Y is a skeleton of Y;
- the set of ramification points of ϕ is contained in Γ_X;
- $\Gamma_X = \phi^{-1}(\Gamma_Y)$ and every vertex of Γ_X is the preimage of a unique vertex of Γ_Y.

[2] Maybe briefly sketch here the ideas of Brewis-Wewers construction

[3] The reader should be aware that in Berkovich's paper, formal schemes locally formally of finite type are called "special"

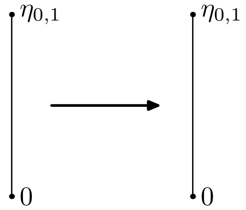
[4] defined as $D^\circ = \{x \in \mathbb{A}_K^{1,\mathrm{an}} : |T(x)| < 1\}$. This shall be introduced in a short introduction about Berkovich spaces where the analytic spectrum, the points $\eta_{a,r}$, types of points in curves, and boundaries are also discussed

Every strictly K-analytic quasi-smooth curve admits a skeleton (cfr. [Duc14, Théorème (5.1.14)]). The existence of skeletons for morphism of quasi-smooth curves is discussed in Bojković paper [Boj16], that generalizes [ABBR15] and [Tem17]. A direct consequence of [Boj16, Theorem 3.0.2] is the following:

Theorem 1.8. *Let $\phi : X \to Y$ be a finite morphism of quasi-smooth K-analytic curves admitting finite graphs Γ_X and Γ_Y as skeletons. Then, there exists a skeleton (Γ'_X, Γ'_Y) of ϕ such that Γ_X is contained in Γ'_X and Γ_Y is contained in Γ'_Y.*

Examples.

- The singleton $\{\eta_{0,1}\}$ is a skeleton for the closed unit disc $D^\bullet = \mathcal{M}(K\{z\})$.
- The Kummer cover $D^\bullet \to D^\bullet$ given by $z \to z^n$ $(n \geq 2)$ has the following skeleton:



- [5]

If Γ_X is a skeleton of X, then every other subgraph $\Gamma'_X \subset X$ containing Γ_X is also a skeleton of X. In this case, we say that Γ'_X is an *enhancement* of Γ_X.
A useful example of enhancement is given by the following proposition, which is a partial analogue of the slope formula [BPR16, Theorem 5.54].

Proposition 1.9. *Let f be a non-constant meromorphic function on X, and let $\Gamma_f \subset X$ be the set of points where the real valued function $-\log|f|$ is not locally constant. Then, Γ_f is a locally finite subgraph of X.*

Proof. Since f is non-constant and meromorphic, the set of its zeroes and poles, that we denote by $D(f)$, is locally finite and consists only of points of type (1). Therefore, it defines a divisor on X, that we denote by $\mathrm{div}(f)$. Moreover, the function $-\log|f| : X \setminus D(f) \to \mathbb{R}$ is piecewise linear on X, and by the non-archimedean Poincaré-Lelong formula [Thu05, Proposition 3.3.15], it has slope different from zero only in the points where $\delta_{\mathrm{div}(f)} \neq 0$ (For a divisor D, recall that δ_D is the extension by linearity of the Dirac measure concentrated on the support of D with value $[\mathcal{H}(x) : K]$ at every $x \in \mathrm{Supp}(D)$). Hence, the set of vertices of Γ_f is also locally finite, and this completes the proof. ☐

It follows from the fact that X retracts by deformation on any of its skeletons that there exists a unique minimal enhancement of Γ_X containing Γ_f. We denote this enhanced skeleton by $\Gamma_{X,f}$.

Example 1.10. For a closed annulus $X = \mathcal{M}(\frac{K\{S,T\}}{ST-a})$, there exists a unique minimal skeleton Γ_X, which is the segment joining the two points of its Shilov boundary, namely $\eta_{0,1}$ and $\eta_{0,|a|}$. Let $f = \sum_{i\in\mathbb{Z}} a_i T^i$ be a non-constant regular function on X and let $Z = \{x \in X : |f(x)| = 0\}$ be the set of zeroes of f. Then, $\mathrm{div}(f) = \sum_{x\in Z}[\mathcal{H}(x) : K]x$, and $\Gamma_{X,f}$ is the unique connected subtree of X that has $Z \cup \{\eta_{0,1}, \eta_{0,|a|}\}$ as set of leaves. The function f is invertible if and only if $Z = \emptyset$, that is, precisely when $\Gamma_{X,f} = \Gamma_X$. At the same time, we know that f is invertible if and only if its Newton polygon is a segment. This is one of several examples of the relationship between the theory of skeletons and Newton polygons, a deep correspondence whose investigation is beyond the scope of the present paper.

[5] add example of equicharacteristic lifting of a Katz-Gabber cover to $R = k[[\varpi]]$

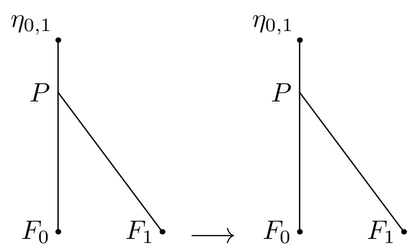
In order to apply the theory of skeletons to Hurwitz trees, we need to introduce metric structures. This is easily done thanks to the construction of the *skeletal metric* described in [BPR13, §5.3], and we just need to transpose this notion in the setting of skeletons of quasi-smooth curves over a complete discrete valuation field. From definition 1.7 follows that the endpoints of an edge $e \in E(\Gamma_X)$ are either both of type (2) or one is of type (1) and the other is of type (2). In the first case, one can show that the preimage of the edge e of the retraction $X \to \Gamma_X$ is a virtual open annulus, i.e. a space X_e defined over a finite separable extension of K, the *field of constants* $\mathfrak{s}(X_e)$ defined by Ducros in [Duc14, (3.1.1.4.)], such that there exists an element $a \in \widehat{\overline{K}}$ with $X_e \otimes_{\mathfrak{s}(X_e)} \widehat{\overline{K}} \cong \mathcal{M}(\widehat{\overline{K}}\{S, T\}/ST - a)$. This element is far from being unique, but its norm $|a| \in \mathbb{Q} \cap]0, 1[$ is, so that we can set the length of e to be $\ell(e) = -\log |a|$. If one of the endpoints of e is of type (1), we set the length of e to be ∞. This construction gives a unique length function $\epsilon : E(\Gamma_X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ that makes Γ_X into a metrized graph.

Definition 1.11. Let $\Lambda = (R[[T]], G)$ be a local action in characteristic zero. The *metric realization* of the Hurwitz tree attached to Λ is the metrized minimal skeleton of the induced G-cover $p_\Lambda : D_1^\circ \to D_2^\circ$, where D_1° is the generic fiber of $\mathrm{Spf}(R[[T]])$ and D_2° is the generic fiber of $\mathrm{Spf}(R[[T]]^G)$.[6]

Let $\Gamma_\Lambda = (\Gamma_1, \Gamma_2)$ be the metric realization of the Hurwitz tree attached to $\Lambda = (R[[T]], G)$. Since the cover p_Λ is Galois, then one has that $\Gamma_2 = \Gamma_1/G$. In what follows, we often identify Γ_Λ simply with Γ_1 to simplify notations.

Remark 1.12. Every vertex $v \in \Gamma_\Lambda$ of the metric realization of a Hurwitz trees is a type 2 point of D_1°. Since we suppose that all the ramification points of p_Λ are K-rational, we have that $v = \eta_{a,\rho}$ for $a \in K$ and $\rho \in |K^\times|$. This allows to associate canonically with v the closed disc $D_v = \{x \in D_1^\circ : |x - a| \leq \rho\}$. For every edge $e \in E(\Gamma_\Lambda)$ originating in v, we have a unique open disc inside D_v that contains e. We call this disc D_e and we associate with it the stabilizer $G_e = \{\sigma \in G : \sigma(D_e) = D_e\}$, which we implicitly consider part of the metric realization of a Hurwitz tree. The length of e can be easily computed as $\ell(e) = \log |\rho_w| - \log |\rho_v|$, where w is the other endpoint of e, and ρ_v (resp. ρ_w) is the radius of D_v (resp. D_w).

Example 1.13. Let ζ_3 be a primitive third root of unity and let σ be the automorphism of $\mathbb{Z}_3^{ur}(\zeta_3, \sqrt{3})[[T]]$ defined by $T \mapsto \frac{T-3}{T-2}$. It has order 3 and on the special fiber it reduces to the automorphism $\bar{\sigma} : \bar{\mathbb{F}}_p[[t]] \to \bar{\mathbb{F}}_p[[t]]$ given by $t \mapsto \frac{t}{t+1}$. Solving the equation $T = \frac{T-3}{T-2}$ gives two distinct fixed points, $F_0 = \frac{3+\sqrt{-3}}{2}$ and $F_1 = \frac{3-\sqrt{-3}}{2}$. Since we have $v(F_1 - F_0) = \frac{1}{2}$, then the Hurwitz tree is a pair of identical skeletons, each one of which has two vertices $\eta_{0,1}, \eta_{0,\frac{1}{\sqrt{3}}}$ and three edges e_1, e_2, e_3:



The length function is such that $\epsilon(e_1) = 2; \epsilon(e_2) = \epsilon(e_3) = \infty$.

This analytic realization of a Hurwitz tree bears similarity with other analogue constructions, but presents two main advantages: the first is that to each vertex of a Hurwitz tree is associated

[6] These are to be intended as Berkovich generic fibers of special algebras. Add a section to explain this.

the residue field of the corresponding point of D_i°. This provides additional algebraic information encoded in this combinatorial object. The second, is that one can enhance the Hurwitz tree and get richer skeletons, that can improve our understanding of the cover p_Λ. As an example of the usefulness of this approach, let us prove a result on the interplay between the shape of Hurwitz trees and ramification, exploiting the existence in Berkovich context of a different function and of a Riemann-Hurwitz type formula, as established by Cohen–Temkin–Trushin in [CTT16]. Recall that a vertex v of an oriented tree is called *terminal* if every successor vertex of v is maximal.

Proposition 1.14. *Let v be a vertex in the metric realization of Γ_Λ, the Hurwitz tree associated with a local action in characteristic zero $\Lambda = (R[[T]], G)$. If there is a successor vertex $w \succ v$ such that:*

1. *w is maximal for the ordering on $V(\Gamma_\Lambda)$;*
2. *for every vertex $z \succ v$, the inertia group G_z is equal to G;*[7]

then v is terminal.

Proof. By maximality of w, the cover p_Λ restricts on a cover of discs $p_{\Lambda,w} : D_w \to D_{w'}$ with a unique ramification point. Since R is a ring of mixed characteristic, then G_w is cyclic and $p_{\Lambda,w}$ is a Kummer cover of order $|G_w|$.[8] The fact that $p_{\Lambda,w}$ is Kummer immediately implies that the slope of the Cohen–Temkin–Trushin different function δ_Λ associated with p_Λ on the edge with endpoints v and w is zero. In other words, the absolute value of the different of the extension $\mathcal{H}(v)/\mathcal{H}(p_\Lambda(v))$ is equal to $|\deg(p_{\Lambda,w})|$, which is also $|\deg(p_\Lambda)|$ by assumption 2. Since in mixed characteristic the absolute value of the different can not be smaller than the valuation of the degree of the associated function, we deduce that δ_Λ is minimal at v. In particular all the slopes of the function δ_Λ at the edges having v as an endpoint are positive.
To prove the proposition, we suppose by contradiction that there is a non-maximal vertex z such that $z \succ v$, and consider the edge e originating in v and terminating in z. By looking at the *local Riemann–Hurwitz formula* [CTT16, Theorem 4.5.4] at the vertex z, we have:

$$-2 = -2|G| + (|G| - 1 - sl_e\delta_\Lambda) + \sum_b (|G| - 1 - sl_b\delta_\Lambda),$$

where $sl_e\delta_\Lambda$ is the slope of the different function on the edge $-e$ (i.e. e with reversed orientation), and b runs over all the other edges of the Hurwitz tree originating in z. By minimality of δ_Λ on v, $sl_e\delta_\Lambda \leq 0$; by non-maximality of z, there are at least two edges of Γ_Λ originating in z. Then, at least one of these edges is such that the slope of the different function on it is positive. By repeatedly applying local Riemann–Hurwitz formulas to the other endpoint of edges with positive slope, we get to the situation where we have a terminal vertex with $m \geq 2$ maximal successor vertices. We can then apply local Riemann–Hurwitz one last time at this terminal vertex to get

$$-2 = -2|G| + (|G| - 1 - sl_{e'}\delta_\Lambda) + m(|G| - 1),$$

where $sl_{e'}\delta_\Lambda$ is the slope of the edge e' connecting the terminal vertex with its unique direct predecessor. In fact we know that the slopes at maximal edges are always zero by what we showed in the first part of the proof. Now, since $sl_{e'}\delta_\Lambda$ is negative by assumption, we find that $m \leq 1$, which is a contradiction. Then z must be maximal, and the same argument applies to every successor of v. □

[7]There should most likely be a counter-example to this result when $G = \mathbb{Z}/p^2\mathbb{Z}$ and $G_w = \mathbb{Z}/p\mathbb{Z}$ provided by an explicit order p^2 cover.

[8]It follows from the techniques of proof of [Col87, Proposition 13], but it is hard to find a direct reference for this: maybe it should be written down as a separate lemma

If $G = \mathbb{Z}/p\mathbb{Z}$, then condition 2 is always satisfied, and Proposition 1.14 is equivalent to a well known fact about Hurwitz trees (shown by Green-Matignon in [GM99, Proposition 1.2.]).

Remark 1.15. This proof can not be adapted in the equicharacteric p case[9].

1.2.2 Piecewise linear functions and the ramification theoretical realization

The proof of Proposition 1.14 makes a crucial use of ramification theory, as it relies on the different function and a formula of Riemann–Hurwitz type. In this section, we define ramification theoretical invariants that can be attached to the metric realization of an Hurwitz tree, in order to make explicit the relationship between the different function and the ($\mathbb{Q}$-valued) characters appearing in Definition 1.6. This construction yields a "ramification theoretical" realization of a Hurwitz tree associated with a a local action $\Lambda = (R[[T]], G)$. In the rest of the section, we denote by $p_\Lambda : D_1^\circ \to D_2^\circ$ the analytic morphism of open discs induced by Λ.

Artin and depth functions on D°

Let $\sigma \in G$ be an automorphism of finite order of $R[[T]]$. The function

$$\begin{aligned} \delta_\sigma : D_1^\circ &\longrightarrow \mathbb{R} \\ |\cdot|_x &\longmapsto -\log|\sigma(T) - T|_x \end{aligned}$$

is called the *depth function* associated with σ. It is piecewise affine when restricted to every open subset of D_1° homeomorphic to an interval of $\mathbb{R}$. This follows easily from the fact that $\sigma(T) - T$ is a regular function on D_1°, by realizing such intervals as skeletons of open annuli and pointed discs and computing the Newton polygon of $\sigma(T) - T$ on them (cf. [Thu05, Proposition 2.2.24]). For every point $x \in D_1^\circ$ there is a unique interval having as oriented endpoints $\eta_{0,1}$ and x.[10] If we denote by $\vec{\partial}\delta_\sigma(x)$ the left derivative at the point x of the depth function restricted to this interval, then the assignment

$$\begin{aligned} a_\sigma : D_1^\circ &\longrightarrow \mathbb{Z} \\ x &\longmapsto \vec{\partial}\delta_\sigma(x) \end{aligned}$$

defines a function that we call the *Artin function* associated with σ.

Theorem 1.16. *Let Γ_Λ be the metric realization of the Hurwitz tree associated with Λ, and let $\{a_e, \delta_v\}_{e\in E(\Gamma_\Lambda), v\in V(\Gamma_\Lambda)}$ be the Artin and depth characters defined by Brewis and Wewers in [BW09, §3]. Then for every $\sigma \in G \setminus \{id\}$ we have the following:*

1. *The Artin function a_σ vanishes outside Γ_Λ, is constant on any edge $e \in E(\Gamma_\Lambda)$, and for such an edge we have*
$$a_e(\sigma) = -a_\sigma(e).$$
2. *The depth function is locally constant outside Γ_Λ, and for every vertex of Γ_Λ we have*
$$\delta_v(\sigma) = |G_v|\delta_\sigma(v).$$

As a result, the assignments $\sigma \mapsto a_\sigma(e)$ and $\sigma \mapsto \delta_\sigma(v)$ are class functions defining characters of G and satisfying the conditions of Definition 1.6.

[9] Add a counterexample here

[10] It should be clear what $\eta_{0,1}$ represents from previous discussions about boundary points

Proof. Let $U \subset D_1^\circ$ be an open subset such that $U \cap \Gamma_\Lambda = \emptyset$. As Γ_Λ contains by definition all the fixed points for the action of G, the regular function $\sigma(T) - T$ has no zeroes in U, and therefore its restriction to U is invertible. From this we get that δ_σ is locally constant outside Γ_Λ and equivalently that $a_\sigma(x) = 0$ for every $x \notin \Gamma_\Lambda$.
After [BW09, Definition 3.1] and the construction of a Hurwitz tree associated to a local action, the depth character at a vertex $v \in V(\Gamma_\Lambda)$ is given by $\delta_v(\sigma) = |G_v| \cdot \log |\eta_v(\sigma(T) - T)| = |G_v|\delta_\sigma(v)$ if σ represents an element of the conjugacy class $[G_v]$, and $\delta_v(\sigma) = 0$ otherwise. For every element σ not representing a class in $[G_v]$, this does not fix the closed disc corresponding to v in D°. Then we have $|\sigma(T) - T|_v = 1$, that is $|G_v|\delta_\sigma(v) = 0 = \delta_v(\sigma)$ also in this second case.
After [BW09, Definition 3.5], the Artin character is given by $-a_e(\sigma) = \#_e(\sigma(T) - T)$, the number of rigid points fixed by σ contained in the open disc associated to e. This number is also, by a Newton polygon argument, the slope of the piecewise affine function $\sigma(T) - T$ on the edge e, which is precisely the definition of $a_\sigma(e)$. □

Remark 1.17. One of the corollaries of Theorem 1.16 is the formula [BW09, Proposition 3.8], that allows Brewis and Wewers to show that the Artin and depth character satisfy property (HT3) of the definition of a Hurwitz tree. In our language, this simply means that, for every $\sigma \in G \setminus \{id\}$ and two consecutive vertices $v, v' \in \Gamma_\Lambda$, the number $\delta_\sigma(v) - \delta_\sigma(v')$ is the slope times the length of the edge joining v and v', which is clearly true by piecewise affinity. We consider this a first evidence of the usefulness of having a piecewise affine structure on our ramification data.

Definition 1.18. Let $\Lambda = (R[[T]], G)$ be a local action in characteristic zero. The *ramification theoretical realization* of the Hurwitz tree attached to Λ is the metric realization Γ_Λ of the Hurwitz tree associated with Λ, endowed with the restrictions to Γ_Λ of the Artin and depth functions a_σ, δ_σ for every element $\sigma \in G$.

Remark 1.19. Let $v \in V(\Gamma_\Lambda)$ be a vertex and $e \in E(\Gamma_\Lambda)$ be an edge of the Hurwitz tree starting in v. The pair (v, e) defines a rank-2 valuation on $R[[T]]$, and therefore a point in the adic open unit disc in the sense of Huber, classically called *of type (5)*. For a non-trivial $\sigma \in G$, the pair $(\delta_\sigma(v), a_\sigma(e))$ can be retrieved by evaluating $\sigma(T) - T$ in this type (5) point, and the Artin function is then associated to the branch (or direction) determined by e, which is consistent with the definition given here in terms of derivative of the depth function.

The Artin and depth characters are tightly related to invariants that appear in Kato's theory of Swan conductors, as exposed in Brewis' PhD thesis [Bre09]. In the same spirit, the Artin and depth functions are related both to Kato's different and to Temkin's theory of wild ramification of Berkovich curves by the following result.

Proposition 1.20. *Let $p_\Lambda : D_1^\circ \to D_2^\circ$ be the finite morphism induced by a local action in mixed characteristic $\Lambda = (R[[T]], G)$. Then the restriction of the Cohen-Temkin-Trushin different function on the Hurwitz tree produces a function $\delta_\Lambda : \Gamma_\Lambda \to [0, 1]$ that can be expressed as*

$$\delta_\Lambda(y) = \prod_{\sigma \in G_y \setminus \{1\}} \delta_\sigma(y)$$

for every $y \in \Gamma_\Lambda$ which is not a terminal vertex, where $G_y \subset G$ is the stabilizer of y.

Proof. Let $y \in D_1^\circ$ of type 2 or 3, and call $x = p_\Lambda(y) \in D_2^\circ$. Since we suppose that the ramification points of p_λ are all K-rational, then there are no non-trivial algebraic extensions of K contained in $\mathcal{H}(y)$ whenever $y \in \Gamma_\Lambda$. Hence, the extension $\mathcal{H}(y)|\mathcal{H}(x)$ has all the properties of an extension of 1-dimensional analytic K-fields as treated in [CTT16] even if K is not algebraically closed, and the different function is well defined. Moreover, $\mathcal{H}(y)|\mathcal{H}(x)$ is G_y-Galois and we can fix a parameter $t_y \in \mathcal{H}(y)$ such that $\mathcal{H}(y) = \mathcal{H}(x)[t_y]$. If P is the minimal polynomial of t_y over $\mathcal{H}(x)$, then $P'(t_y)$ generates the annihilator of the module of relative differentials $\Omega_{\mathcal{H}(y)^\circ|\mathcal{H}(x)^\circ}$ (cf. [GR03, Claim 6.3.22]). Then we can express the different as

$$\delta_\Lambda(y) = |P'(t_y)| = \prod_{\sigma\in G_y\setminus\{1\}} |t_y - \sigma(t_y)| = \prod_{\sigma\in G_y\setminus\{1\}} \delta_\sigma(y),$$

and the result follows. □

Kato's theory of different is actually richer, and consists also of a differential part, encoding more information than that described by the depth function. We exploit this higher complexity in section 1.2.4, allowing us to rephrase the local lifting problem in terms of Berkovich-Hurwitz trees. However, already the non-differential part of the ramification invariants can lead to interesting results. To illustrate this, we give a proof of a classical theorem on local actions [GM99, Theorem 3.1.] in a purely analytic fashion.

Theorem 1.21. *Let $\Lambda = (R[[T]], \mathbb{Z}/p\mathbb{Z})$ be a local action with $m+1$ rigid ramification points. If $m < p$, then the Hurwitz tree is* elementary*, that is, it has $m+1$ edges and a unique vertex of degree > 1.*

Proof. Let $p_\Lambda : D_1^\circ \to D_2^\circ$ denote, as usual, the cover of open discs associated to Λ. The slope of the different function δ_Λ on the Hurwitz tree $\Gamma_\Lambda \subset D_1^\circ$ can be easily deduced from the number of ramification points in the following way. If e is a terminal edge, then the cover restricted to the corresponding pointed disc is Kummer, and $sl_\delta(e) = 0$. If e is any other edge, then we have (applying local RH formula) $sl_\delta(e) = -k(p-1)$, where k is the number of terminal edges t such that $t \succ e$. If Γ_Λ is not elementary, this means that the different function decreases from 1 to $|p|$ with at least two successive distinct slopes $m(p-1)$, and $m'(p-1)$ on two non-terminal edges that we denote by e and e'. After choosing coordinates T_1 and T_2 for D_1° and D_2° respectively, we can translate these slopes to a statement on the coefficients of the presentation associated with p_Λ in the following way. Thanks to the classification of étale annular p-covers [BT17, Theorem 4.3.8], on the open annulus whose skeleton is e we can write

$$T_2 = T_1^p + cT_1^n, \text{ with } n = m(p-1) + p.$$

This occurs because the slope of the different can be written also as $p - n$, so that we have $n = m(p-1) + p$. Moreover, the restriction of p_Λ on the open annulus whose skeleton is e' has a presentation of the form

$$c'T_2' = \sum_i a_i {T_1'}^i = (rT_1' + c)^p + c(rT_1' + c)^n,$$

Now, the dominant term of the derivative of the series on the right hand side must be of degree $n' = m'(p-1) + p > p$, so that we can compute it by looking only at the terms of degree greater than p in

$$c(rT_1' + c)^n = \sum_{k=0}^{n} \binom{n}{k} c^{n-k+1} r^k {T_1'}^k.$$

The condition that $m < p$ is equivalent to $n < p^2$, and n is not divisible by p. Hence the binomial coefficient $\binom{n}{p+1}$ is not divisible by p and then the degree $p+1$ term becomes dominant in the derivative. But this would imply that $n' = p+1$ which is not possible, as it would imply $p = 2$ and $m' = 1$, and then the only possibility is that Γ_Λ is elementary. □

Remark 1.22. In the language of [BT17], we can prove that for $m < p$ equidistance is the only possibility also by showing that there is not a possible p-enhancement associated with a non elementary Hurwitz tree having $m+1 < p+1$ terminal vertices. This is equivalent to prove the non-existence of certain bivariant exact differential forms, and hence boils down to the original computations by Green and Matignon.

1.2.3 Hurwitz subtrees

Given a Hurwitz tree associated with a local action in characteristic zero, we can consider its proper subtrees. We show that they can be associated with the ramification theoretical realization of a Hurwitz tree corresponding to other local actions, determined uniquely from the first action and the shape of the tree. From this we deduce obstructions to the realization of certain Hurwitz data as coming from local actions.
Let $(\Gamma_\Lambda, \{a_\sigma\}, \{\delta_\sigma\})$ be the ramification theoretical realization of a Hurwitz tree coming from a local action $\Lambda = (R[[T]], G)$. Recall that the existence of a metric realization in the Berkovich context yields a canonical association of any vertex $v \in V(\Gamma_\Lambda)$ with a closed disc $D_v \subset D_1^\circ$, and of any edge $e \in E(\Gamma_\Lambda)$ with an open disc $D_e \subset D_1^\circ$. All these discs are centered in a K-rational point and their radius belongs to $|K^\times|$. It is natural for an edge e originating in a vertex v to restrict the action of G_e to D_e, and in this way get a local action $\Lambda_e = (R[[T']], G_e)$. It is immediate from the definitions that the ramification theoretical realization of Γ_{Λ_e} consists of the metric realization of the subtree of Γ_Λ rooted in v with root-edge e, endowed with the restriction of the Artin and depth functions to Γ_{Λ_e}.

In this section we discuss how the shape of an Hurwitz tree is related to the shape of certain subtrees. In order to do so, let us fix a pair (v, G_v) consisting of a vertex of Γ_Λ and the stabilizer of the associated disc D_v. The action of G_v can be reduced modulo the uniformizer of R to give rise to an automorphism of the residual curve at v that fixes the point at infinity. In this way, we get an homomorphism $G_v \to \mathrm{Aut}(\mathbb{A}^1_k)$. [11]

Lemma 1.23. *Let $\bar{\sigma} \in \mathrm{Aut}(k[t])$ be an automorphism of finite order. Then one of the following is verified:*

- $\bar{\sigma}(t) = t$
- $\bar{\sigma}(t) = \zeta_n \cdot t$, *with ζ_n primitive n-th root of unity and $(n,p) = 1$*
- $\bar{\sigma}(t) = t + b$ *with $b \in k$*

or a composition of these.

Proof. We have $\bar{\sigma}(t) = at + b$ with $a \neq 0$. Let $n \in \mathbb{N}$ be such that $t = \bar{\sigma}^n(t) = a^n t + \sum_{i=0}^{n-1} a^i b$. This implies $a^n = 1$. Then either $a = 1$ or $a = \zeta_n$, which is possible only when $(n,p) = 1$. In the first case the condition above becomes $t = t + n \cdot b$ giving $n \cdot b = 0$. When $b = 0$ we get the identity and when $b \neq 0$ we get a translation. In the second case $a = \zeta_n$ and $\bar{\sigma}$ is a rotation of order prime to p, possibly composed with a translation of order p. ☐

Corollary 1.24. *Let (v, G_v) and $\beta : G_v \to \mathrm{Aut}(\mathbb{A}^1_k)$ as above. If $\sigma \in G_v$ has order divided by p and $\beta(\sigma)$ fixes a closed point of $\mathbb{A}^1_k$, then $\beta(\sigma) = \mathrm{id}$.*

[11] There is a precise definition of a tangent space of D at v that is homeomorphic via Riemann-Zariski spaces to the residual curve. We address this theory in the part of this paper describing global Hurwitz graphs.

This corollary implies in particular that, when the local action Λ is wildly ramified and has at least a fixed point, then the Hurwitz tree is strictly contained in the Berkovich ramification locus of p_Λ, which is in this case an infinite tree. This is well known by experts in wild ramification of Berkovich curves (see [Fab13] and [Tem17]), but the determination of the Berkovich ramification locus of p_Λ remains a much more difficult task than to compute the Hurwitz tree of Λ.

Corollary 1.25. *Let Γ_Λ be an Hurwitz tree coming from a local action in characteristic zero, and let v be a vertex of Γ_Λ. If G_v is a p-group, then the subgroup $G_{v'} \subset G_v$ does not depend on the choice of a successor $v' \succ v$. Moreover, for every $\sigma \in G_v$, one has $\sigma^p \in G_{v'}$.*

Proof. We have that $G_{v'} = G_e$ where e is the edge joining v and v'. Hence $G_{v'} = \{\sigma \in G_v : \sigma(e) = e\}$. The identification between the space of edges originating at v and closed points of $\mathbb{A}^1_k$ yields that $\sigma \in G_v$ is in $G_{v'}$ if and only if $\beta(\sigma) = \mathrm{id}$. By Corollary 1.24, $G_{v'}$ coincides with $G_{v''}$ for any other choice of a successor vertex $v'' \succ v$. By Lemma 1.23, if $\beta(\sigma) \neq \mathrm{id}$ then $\beta(\sigma)$ is of order p. Hence $\sigma^p \in G_{v'}$. □

[12]

1.2.4 The Swan bundle and the differential realization

Potential theory and harmonic functions can be defined on non-archimedean analytic curves, thanks to the works of Thuillier, and Baker-Rumely. In this section, we study certain analytic vector bundles that naturally arise from differentials of harmonic functions. Certain global sections of these bundles can be associated with local actions in order to define ramification invariants that refine the Artin and depth functions.

Definition 1.26. Let S be a finite polyhedron and Γ a locally finite set of S. A m-harmonic function h over $S \setminus \Gamma$ is a function that is harmonic over S minus its vertices and satisfies, for every vertex v, the condition

$$\sum_{a \in A(v)} \lambda_a(h) = -m.$$

The set of all such functions is denoted by $\partial H(S, \Gamma)_m$.

Notice that $\partial H(S, \Gamma)_0$ is the usual vector space of harmonic functions.
Moreover the m defines a structure of graded $\mathbb{R}$-vector space on $\partial H(S, \Gamma) = \bigoplus_m \partial H(S, \Gamma)_m$.
Let σ be an automorphism of order p^n of the Berkovich open unit disc Y. Then $f_\sigma := \sigma(T) - T \in \mathcal{O}_Y$ is a regular function. Let $Y := \mathcal{M}(K\{z\})$ be the Berkovich unit disc and $\sigma \in \mathrm{Aut}_R(R\{z\})$ an automorphism of finite order. Then it induces by functoriality an homeomorphism $\Sigma : Y \to Y$ given by $\Sigma(x) = x \circ \sigma$. With straightforward calculations we can see that $\Sigma(\eta_{a,\rho}) = \eta_{a(\sigma(z)),\rho}$, which implies the following easy but important fact.

Lemma 1.27. *$\eta_{a,\rho}$ is a fixed point for $\Sigma \Leftrightarrow \Sigma(D(a, R]) = D(a, R]$, inducing an homeomorphism of Berkovich discs.*

Having an automorphism of finite order of the open disc $\sigma \in \mathrm{Aut}_R(R[[z]])$, is a weaker condition. Nevertheless it still induces an homeomorphism of the open Berkovich disc $Y^\circ = U_{\rho<1} D(0, \rho]$. Being this function continuous, it can be extended to an homeomorphism of the topological closure $Y^\circ \cup \{\eta_{0,1}\}$.

Remark 1.28. Explicit calculations show that $|f_\sigma(z)^{p^n}|_K < 1$ whenever $\mathrm{ord}(\sigma)|p^n$. It would be useful to say something more about the correspondence $\sigma \mapsto f_\sigma$ mod π_R.

[12] Insert here: a remark about Hurwitz trees of wild actions without fixed points.

We can wonder what is the set of fixed points by Σ like and if it is interesting to study the variation in absolute value of some functions on it. Look at $\sigma(z) - z \in R[[z]]$ as an analytic function over the open unit disc. The values of this functions at rigid points tell us something about the Berkovich points fixed by Σ.

Lemma 1.29. *Consider a rigid point $a \in \mathfrak{m}_R$. The value $\rho(a) := |a(\sigma(z) - z)|$ is exactly the radius of the smallest rigid disc fixed by Σ and centered in a.*

Proof. The homeomorphism Σ fixes the point $\eta_{a,\rho(a)}$ since $\Sigma(\eta_{a,\rho(a)}) = \eta_{a(\sigma(z)),\rho(a)}$ and we know by hypothesis that $|a(\sigma(z) - z)| \leq \rho(a)$, then $\eta_{a(\sigma(z)),\rho(a)} = \eta_{a,\rho(a)}$.
Every disc $\eta_{a,\rho} < \eta_{a,\rho(a)}$ is not fixed, otherwise $\Sigma(\eta_{a,\rho}) = \eta_{a,\rho}$ would imply $\eta_{a(\sigma(z)),\rho} = \eta_{a,\rho}$ and then $|a(\sigma(z) - z)| \leq \rho < \rho(a)$ giving a contradiction. □

The Weierstrass preparation theorem gives us the equality $\sigma(z) - z = \pi^n \cdot u(z) \cdot P(z)$ with $u(z)$ unit in $R[[z]]$ and $P(z) \in R[z]$ polynomial of degree m which reduces to $z^m \in k[z]$. it is therefore easy to calculate $|a(\sigma(z) - z)|$ which is $p^{-n \cdot v_R(P(a))}$. For example when considering a disc without fixed points (and every case reduces to this easy example) the polynomial $P(z)$ is a constant and $|a(\sigma(z) - z)| = \pi^{-n}$. The fixed rigid discs are then all the $\eta_{0,\rho}$ with $\eta > \pi^{-n}$. This set clearly contains $\eta_{0,1}$ which is the Hurwitz tree (we allow in fact Hurwitz trees to exist also in case of no fixed rigid point), but is much bigger than that (it is indeed an "infinite graph").

Definition 1.30. Let Λ be a local action in characteristic zero. Then, the *Swan bundle* Ω_Λ on D_1° is defined as the line bundle $\Omega(\log(Ram))$ endowed with the model metric coming from the stably marked model.

In fact the metric tree is exactly given by the skeleton of the analytic space $D(0,1] - (\{\eta_{0,1}\} \cup \Delta)$, the groups G_v are monodromy groups along subsets of Δ and the Artin and depth characters are given by the valuation of analytic functions f_σ that take their zeroes in Δ_σ. This is a classic problem in the complex analytic setting: we study the covering space $X \to \mathbb{C} - \{P_1, \dots, P_n\}$ given by a logarithmic differential form $\omega = \frac{df}{f}$ throughout its monodromy groups. These give values of the integral $\int_{D(P,\rho]} f(z)dz$ around discs centered in fixed points and therefore informations on the cover.
The use of rigid geometry in the study of lifting problem for coverings has in fact turned out to be very fruitful, but some features that appear in its use seems to be quite obscure. Showing that the valuations occurring in the definition of combinatorial datas are actually point of the analytic space on which we can evaluate our function helps us to clarify the setting.

Theorem 1.31. *Let Λ be a local action in characteristic zero of associated group $G = \mathbb{Z}/p\mathbb{Z}$. Then there is a section $\omega \in \Omega_\Lambda$ whose reduction at every vertex of Γ_Λ yields a good deformation datum.*[13]

Proof. Let $\omega = \frac{dg}{g}$. There is a bijection between

$$\{\text{domains on which } \Omega^1_Y(\log D) \text{ is trivial }\} \quad \text{and} \quad \{ \text{ edges of } \Gamma(D)\}$$

while for each vertex we have the cocycle relation (1.1).
Fix a vertex v of $\Gamma(D)$. It is a point of type (2), and the reduction of the germ Y_v is homeomorphic to the affine line $\mathbb{A}^1_k$ if $x = \eta_{0,1}$ and to the projective line $\mathbb{P}^1_k$ otherwise.
The section ω has a germ $\omega_v \in \Omega^1_{Y,v}(\log D)$ in an analytic neighborhood of v. The corresponding reduction $\tilde{\omega}_v \in \Omega_{k(z)}$ is a differential form on the residual curve of v.

[13] In the sense of Henrio, or Brezner-Temkin

The form ω_v is logarithmic at every v and, by reduction of coefficients, $\tilde{\omega}_v$ is logarithmic too. We have to show that $\tilde{\omega}_v$ is exact on every additive vertex v. This is true by residue theorem in positive characteristic.
Finally the condition of having no zeroes in the open unit disc implies that $\tilde{\omega}_v$ has zeroes just in infinity for every v, and hence it is a good deformation datum.
This collection of good deformation data on each vertex of $\Gamma(D)$ gives rise to the lifting (see [BWZ09]) and the fact that $\Gamma(D)$ is the Hurwitz tree of the lifting is true by construction. □

1.3 Applications to models of torsors

1.3.1 An interpretation of the stably marked model

The action of σ over boundaries can be studied over rings of the form $R[[z]]\{z^{-1}\}$. From the Berkovich point of view these rings are quite natural: if we compactify the Berkovich open unit disc we are just adding one point (the Gauss point, corresponding to the Gauss valuation $(a, 1)$). It is natural to consider as local functions over this point the functions which are uniformly bounded (on the closed disc) and the construction of the ring $R[[z]]\{z^{-1}\}$ as completion of $R[[z]]$ for the Gauss valuation shows that this is right the ring we are looking for.

1.3.2 Combinatorial conditions

Definition 1.32. Let n be a positive natural number, $(m_1, \ldots, m_n)\in \mathbb{N}^n$ and Y be the Berkovich unit disc and let f be a regular function over Y. We say that f is of type $(m_1, \ldots, m_n)$ if $(f + z)^{p^{i-1}} - z$ has exactly $m_i + 1$ zeroes for $i = 1, \ldots, n$.

Example 1.33. Let σ be an automorphism of $R[[z]]$ and let m_i be the cardinality of the set $\{x \in \mathrm{M}_0(R[[z]]) : \Sigma^{p^i}(x) = x\}$. Then $\sigma(z) - z$ is a regular function over Y of type $(m_1, \ldots, m_n)$ [for any choice of n].

Let $x \in Y$ be a point of type (2) and let f be a regular function over Y. Call $Z(f)$ the set of zeroes of f, and C the curve over k which is the special fiber of a semi-stable model of $Y \setminus Z(f)$. To f can be associated a divisor over the residual curve C_x in the following way:

$$\mathrm{div}_x(f) = \sum_{t\in T_xS} \lambda_{x,t}(f)[\tilde{x}_t].$$

[14]

1.4 Deformation data beyond the order p case

We want to study, and possibly parametrize, some deformations of Kummer equations in characteristic zero. The formalism of good deformation data is a relative concrete way to use the informations provided by the differential Swan conductor.

Definition 1.34. Let $G = P \rtimes N$ be a cyclic by p group with character χ, $f \in k(z)$ and $\omega = f dz \in \Omega^1_{\mathbb{P}^1_k}$. We call it a *good deformation datum* with conductor m for G if the following conditions are satisfied:

- ω is logarithmic.

[14] Define $\lambda_{x,t}(f)$ as the slope of the function $\mathrm{dlog}(|f|)$ on the edge t

- There is a faithful action of N on $\Omega^1_{\mathbb{P}^1_k}$ such that
$$\sigma.\omega = \chi(\sigma) \cdot \omega$$
when $\sigma \in N$.
- ω has an unique zero, of order $m-1$.

This definition applies to the lifting problem in the following sense

Proposition 1.35. *Let $\sigma \in \mathrm{Aut}_G(k[[t]])$. If there exist a formal model $\mathcal{X}$ of the unit disc and a differential form $\omega(\sigma) \in \Omega^1_{\mathcal{X}_s}$ such that its restriction on every terminal irreducible component of $\mathcal{X}_s$ is a good deformation datum, then the action of σ lifts to characteristic zero.*

Remark that a good deformation datum matches the differential Swan conductor when restricted to a boundary action

1.4.1 Reduction of germs

We recall here some results that we need to study the reduction of an invertible sheaf on a Berkovich curve. Most of these are originally exposed in the articles [Tem00] and [Tem04].[15]

1.4.2 Differential forms

Modules of differential forms for affinoid algebras are defined (like in algebraic geometry) with derivations: whenever $\mathcal{A}$ is an affinoid algebra over K, then $\Omega^1_{\mathcal{A}}$ is the universal object representing the functor $\mathrm{Der}(\mathcal{A}, \cdot)$. Glueing the affinoid domains we obtain a locally free sheaf of $\mathcal{O}_X$-modules over any analytic space X.
Let $\mathcal{A} = K\{z\}$ and consider Ω^1_Y the sheaf of 1-forms over the Berkovich unit disc. Its global sections are isomorphic to $\Omega^1_{\mathcal{A}} = \mathcal{A}dz$ and hence we can write fdz for every differential form on the disc.

Good deformation datas have at most simple poles. We can restrict ourselves to consider a finite set of points $x_1, \dots, x_n \in R$ and the sheaf $\Omega(\log D)$ of differential forms having at most simple poles in the divisor $D = \sum x_i$. It is a locally free sheaf of rank one, generated by the form $\frac{dg}{g}$ with $g = \prod_{i=1}^n (X - x_i)$.
We can show exactly where does this locally free sheaf trivialize. Call $P(x_i, \rho)$ the punctured disc centered in x_i of radius ρ and $A(x_i, r, R)$ the annulus centered in x_i of big radius R and small radius r.

Proposition 1.36. *Let $D = \sum x_i$ be a divisor on the Berkovich disc, with $x_i \neq x_j$ for every i and j. Then the sheaf $\Omega(\log D)$ is isomorphic to $\mathcal{O}_X$ on every disc not containing any of the x_i, on every $P(x_i, \rho)$ with $\rho < \min\{|x_i - x_j|, i \neq j\}$ and on every $A(x_i, r, R)$ satisfying the condition that do not exist any x_k such that $r < |x_i - x_k| < R$.*

The proposition says that we are able to construct a set of discs and annuli on which the sheaf $\Omega(\log D)$ is trivial. It remains to show which cocycles give the glueing on “intersections”.

1.4.3 Smooth metrics

Let $\mathfrak{L}$ be an invertible sheaf on a K-analytic curve X, A *smooth metric* on $\mathfrak{L}$ is the data, for each open subset of X of a function $-\log(\|\cdot\|) : \Gamma(U, \mathfrak{L}) \to A^0(U)$ such that, if $s \in \mathfrak{L}(U)$ and $s' \in \mathfrak{L}(U')$ are two invertible sections then $s' = fs$ on $U \cap U'$ for some $f \in \mathcal{O}^\times(U \cap U')$ and
$$-\log(\|s'\|) = -\log(\|s\|) - \log(|f|).$$

[15] Introduce here only what is needed in the following.

1.4.4 Reduction of differential forms

Lemma 1.37. *Let $x \in Y$ and $s \in \mathfrak{L}_x$ be a section of a locally free sheaf on Y_x. There is a bijection between analytic subdomains of Y_x on which $\mathfrak{L}_x$ is trivial and open subsets of $\widetilde{Y}_x$ on which $\widetilde{\mathfrak{L}}_x$ trivializes.*
Moreover let V_α be an analytic subdomain of Y_x with a trivialization $\mathfrak{L}_{|V_\alpha} \xrightarrow{\tau} \mathcal{O}_X|V_\alpha$ such that $\tau(s_{|V_\alpha}) = f$. Then $\tilde{\tau}(\tilde{s}) = \tilde{f}$.

Let $v = \eta_{a,\rho}$ be a point of type (2) of Y. Let V be a neighborhood of v, $V_a = D(a,\rho) \cap V$ and $V_\infty = \{|T-a| \geq \rho\} \cap V$. A germ of differential form is defined by the cocycle

$$g_{a,\infty}(s) = -\frac{1}{(z-a)^2} s \tag{1.1}$$

for every section $s \in \Omega^1_{Y,v}(V)$.

1.5 Reformulation of the local lifting problem

Let $\lambda = (G, k[[t]])$ be a local action in positive characteristic. In this section, we introduce the notion of λ-compatibility for functions and metrized differential forms on the unit disc. This notion will be crucial to show that a suitable set of compatible Hurwitz data yields the existence of a lifting of λ to characteristic zero. Conversely, by using the constructions of previous sections, any lifting corresponds to a set of compatible data, giving a way to classify liftings of λ.

Theorem 1.38. *Let G be a finite group generated by g elements, and $\lambda = (G, k[[t]])$ be a local action in characteristic $p > 0$. Then λ lifts to characteristic zero if and only if there exists a metrized bundle $(\Omega, \|\cdot\|)$ over D_1°, and a g-uple of sections $(\omega_1, \dots, \omega_g)$ that are λ-compatible. If this is the case Ω is identified with the Swan bundle of a lifting.*

In order to prove this result, we need some preliminary facts.

Proposition 1.39. *Let $p : Y \to X$ be a lifting of a G-cover in positive characteristic. Then there exists a character $\omega : G \to H^0(Y, \mathfrak{L}\Omega_Y)$ satisfying the conditions of Theorem 1.31, and the minimal A that can be taken is the Hurwitz tree associated with p.*

A characterization of logarithmic differential forms

We have seen how exact and logarithmic forms play an important role to determine liftability of actions. In general is not easy to understand when a meromorphic form is logarithmic. There is a well known characterization of logarithmic differential forms in positive characteristic that can be expressed in the following lemma

Lemma 1.40. *A meromorphic differential form $\omega = g(t)dt$ on $\mathbb{P}^1_k$ is logarithmic if and only if the Weil divisor D of poles of g is such that*

$$\omega \in \Omega^1(\log D) \quad \text{and} \quad \mathrm{Res}_P(\omega) \in \mathbb{F}_p$$

for every $P \in \mathrm{Supp}(D)$.

If we want to find an analogue result over K that reduces to Lemma 1.41 we have to ask for a characterization of logarithmic differential forms over Y.
The Lemma 1.41 implies that a necessary condition is to have at most simple poles with integer residues. Moreover we are allowed to study only the case of regular (i.e. without poles) differential forms. In fact, if f has simple poles in $\{x_i, i = 1, \dots, n\}$ with residues $\{a_i, i = 1, \dots, n\}$, and

$p = \prod_i (T - x_i)^{a_i}$, then the form fdT is logarithmic if and only if the regular form $(f - \frac{dp}{p})dT$ is logarithmic.
Let $\mathcal{U} = 1 + \mathfrak{m}\{T\}$, by Weierstrass preparation theorem we can write every function $g \in K\{T\}$ as a product

$$g(T) = up(T)h(T) \quad \text{with} \quad u \in K^\star,\ p \in R[X] \text{ and } h \in \mathcal{U}$$

so that, in the regular case, we have to look for differential forms that can be written as $\frac{dh}{h}$ with $h \in \mathcal{U}$.

Proposition 1.41. *Let $\omega = fdT \in \Omega^1_Y(Y)$ such that $f = \sum_n a_n T^n$ and $W_n(x_i)$ the n-th Witt polynomial in $\varphi(n)$ variables. Let (u_n) be the sequence recursively defined by*

$$\begin{cases} u_1 = -a_0 \\ u_n = -\frac{1}{n}(a_{n-1} + W_n(u_d)) \end{cases} .$$

Then the following are equivalent:

1. *the differential form ω is logarithmic*
2. *for every $n \in \mathbb{N}$, $u_n \in \mathfrak{m}$ and $\lim_n u_n = 0$.*

Explicitly, when the conditions are satisfied, we can write $\omega = \frac{dh}{h}$ with

$$h = \prod_{n>0} (1 - u_n T^n) \in \mathcal{U}.$$

2 The global picture: Berkovich-Hurwitz graphs

In this section, we make use of several results about skeletons established in Section 1 in order to study the ramification of G-covers of analytifications of smooth projective curves over K.
[16]

[16]Insert here:

- The local-global principle
- Katz-Gabber covers
- Global lifting problem

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