Lecture 2. Berkovich spectra

Recall · A Banach ring is the datum

$$(A, \|\cdot\|)$$

where A is a ring (commutative with unity) and $\|\cdot\|$ a norm on A s.th. A is complete wrt $\|\cdot\|$.

- A valued field is the datum

$$(k, |\cdot|)$$

where k is a field and $|\cdot|$ is a multiplicative norm on k.

- A Banach algebra $(A, \|\cdot\|)$ over a valued field $(k, |\cdot|)$ is a Banach ring that is also a k-algebra and satisfies $\|x\| = |x|$ $\forall x \in k$.

Examples

- $(\mathbb{Z}, |\cdot|_\infty)$ is a Banach ring
- $(\mathbb{C}, |\cdot|_\infty^\varepsilon)$ is a valued field $\forall\ 0 < \varepsilon \leq 1$

[Exercise: what happens if $\varepsilon > 1$?]

- $(\mathbb{Q}_p, |\cdot|_p)$ $(\mathbb{C}((t)), |\cdot|_t)$ are valued fields
- $(A, |\cdot|_0)$ is a Banach ring
- $(\mathbb{C}, \max\{|\cdot|_\infty, |\cdot|_0\})$ is a Banach ring <u>but not a valued field</u>
- Let $(k, |\cdot|)$ be a valued field. Pick $r > 0$.

$$k\{r^{-1}T\} := \left\{ \sum_{i=0}^{\infty} a_i T^i \;/\; a_i \in k, \sum_{i=0}^{\infty} |a_i| r^i < \infty \right\}$$

is a k-Banach algebra with norm

$$\left\| \sum a_i T^i \right\| = \sum_{i=0}^{\infty} |a_i| r^i .$$

(think of this as convergent functions on a closed disc $B(0,r)^+$)

<u>Note</u>: if $|\cdot|$ is non-archimedean, then

$$\sum_{i=0}^{\infty} |a_i| r^i < \infty \iff \lim_{i \to \infty} |a_i| r^i = 0$$

§. Berkovich spectrum

A multiplicative semi-norm on a ring A is a function

$$|\cdot| : A \to \mathbb{R}_{\geq 0} \quad \text{s.th.}$$

- $|0| = 0$, $|1| = 1$
- $|f-g| \leq |f| + |g|$
- $|fg| = |f||g|$

Definition: The Berkovich spectrum of a Banach ring $(A, \|\cdot\|)$ is

$$\mathcal{M}(A) = \left\{ \begin{array}{l} \text{multiplicative semi-norms on } A \\ \text{bounded by } \|\cdot\| \end{array} \right\}$$

endowed with the weakest topology making

$$\begin{aligned} ev_f : \mathcal{M}(A) &\longrightarrow \mathbb{R}_{\geq 0} \\ |\cdot| &\longmapsto |f| \end{aligned}$$

continuous for every f.

Why is this a good definition?

1) it has nice topological properties

Theorem (Berkovich)

- $\mathcal{M}(\mathcal{A})$ is nonempty, compact and Hausdorff.
- $\mathcal{M}(k\{r^{-1}T\})$ is pathwise connected

2) it generalizes complex analytic geometry

Theorem (Gelfand-Mazur)

$\mathbb{C}$ is the only $\mathbb{C}$-Banach algebra that is also a field.

Corollary Let $\mathcal{A}$ be a $\mathbb{C}$-Banach algebra.

Then $\mathcal{M}(\mathcal{A}) \simeq \mathrm{Max}(\mathcal{A}) = \{\text{maximal ideals of } \mathcal{A}\}$

(e.g. $\mathcal{M}(\mathbb{C}\{r^{-1}T\}) \simeq B(0,r)^{+} \subset \mathbb{C}$

etc.)

3) It is analogue to Spec(A).

$$A \text{ ring} \rightsquigarrow \operatorname{Spec}(A) = \{ \mathfrak{p} \lhd A : \mathfrak{p} \text{ prime} \}$$

$$= \{ \chi : A \longrightarrow K \text{ ring homomorphism} \} / \sim$$

$$\begin{tikzcd}
 & K' \\
A \arrow[ur, "\chi'"] \arrow[r] \arrow[dr, "\chi''"'] & K \arrow[u] \arrow[d] \\
 & K''
\end{tikzcd}$$

$\chi' \sim \chi''$

$\chi \longrightarrow \operatorname{Ker}(\chi)$

$$\mathcal{A} \text{ Banach ring} \rightsquigarrow \mathcal{M}(\mathcal{A})$$

$$= \{ \chi : \mathcal{A} \longrightarrow K \text{ bounded Banach ring homomorphism} \} / \sim$$

Proof.

- $\chi \longrightarrow \quad |\cdot|_\chi : \mathcal{A} \to K \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0}$
- $|\cdot|_x : \mathcal{A} \longrightarrow \mathbb{R}_{\geq 0} \quad \mathcal{A} \longrightarrow \mathcal{A}/_{\operatorname{Ker}|\cdot|} \longrightarrow \boxed{\widehat{\operatorname{Frac}\left(\mathcal{A}/_{\operatorname{Ker}|\cdot|}\right)} =: \mathcal{H}(x)}$

important object

Exercise: every $\chi : \mathcal{A} \longrightarrow K$ factors through $\mathcal{A} \to \mathcal{H}(x)$ for $x = |\cdot|_\chi$.

Examples · $\mathcal{M}(k, |\cdot|)$

· $\mathcal{M}((\mathbb{Z}, |\cdot|_\infty))$ Remark: $|\cdot|_\infty$ is the largest possible norm we can put on $\mathbb{Z}$.

φ

$|\cdot|_x$

b/c multiplicative

$\rightsquigarrow \ker |\cdot|_x \in \operatorname{Spec}(\mathbb{Z}) = \{(0)\} \cup \{(p) : p \text{ is prime}\}$

① If $\ker |\cdot|_x = (p)$ then induces a norm on $\mathbb{Z}/_{p\mathbb{Z}} = \mathbb{F}_p$:

exercise: a ring norm induces a unique norm on its quotients.

But $\mathbb{F}_p \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0}$ must be the trivial norm

$$\left(|a|^p = |a| \quad \forall\, a \in \mathbb{F}_p \right)$$

$$\Rightarrow \quad |\cdot|_x = |\cdot|_{p,0} := \begin{cases} x \longmapsto 0 & \text{if } x \in p\mathbb{Z} \\ x \longmapsto 1 & \text{otherwise} \end{cases}$$

② If $\ker |\cdot|_x = (0)$ then this is a norm on $\mathbb{Q}$.

Theorem (Ostrowski)

Any norm $\mathbb{Q} \longrightarrow \mathbb{R}_{\geq 0}$ is one of the following:

$$\begin{cases} - \; |\cdot|_0 & \\ - \; |\cdot|_p^{\varepsilon} & 0 < \varepsilon < +\infty \\ - \; |\cdot|_\infty^{\varepsilon} & 0 < \varepsilon \leq 1 \end{cases}$$

RLH.

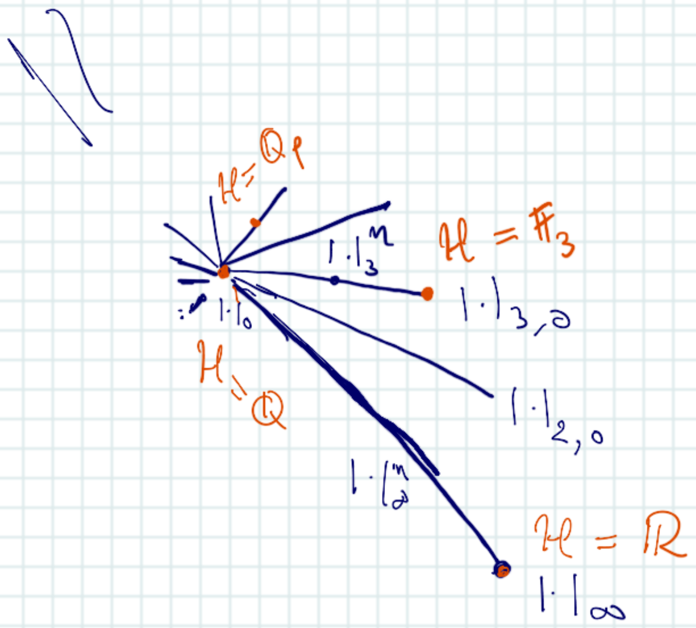
$$[0,1] \longrightarrow \mathcal{M}(\mathbb{Z})$$

The maps
$$\eta \xmapsto{\varphi_p} \begin{cases} |\cdot|_{p,0} & \eta = 0 \\ |\cdot|_p^{-\log(\eta)} & \eta \in (0,1) \\ |\cdot|_0 & \eta = 1 \end{cases}$$

$$\eta \xmapsto{\varphi_\infty} \begin{cases} |\cdot|_0 & \eta = 0 \\ |\cdot|_\infty^{\eta} & \eta \in (0,1] \end{cases}$$

are homeomorphisms onto their images

$\rightsquigarrow$ $\mathcal{M}(\mathbb{Z})$



· $\mathcal{M}(\mathbb{C}, \max\{|\cdot|_\infty, |\cdot|_0\})$

$\overset{\psi}{|\cdot|_x}$ is either $|\cdot|_0$ or $|\cdot|_\infty^\varepsilon$ for $0 < \varepsilon \leq 1$

· $\mathcal{M}(\mathcal{K}\{T\})$

$f \in \mathcal{K}\{T\}$ $\|f\| = \max_i \{|a_i| r^i\}$

$'' \sum a_i T^i$: $|a_i| \to 0$

examples of points:

$a \in \mathcal{K} \rightsquigarrow \eta_{a,0} : f \longmapsto |f(a)|$

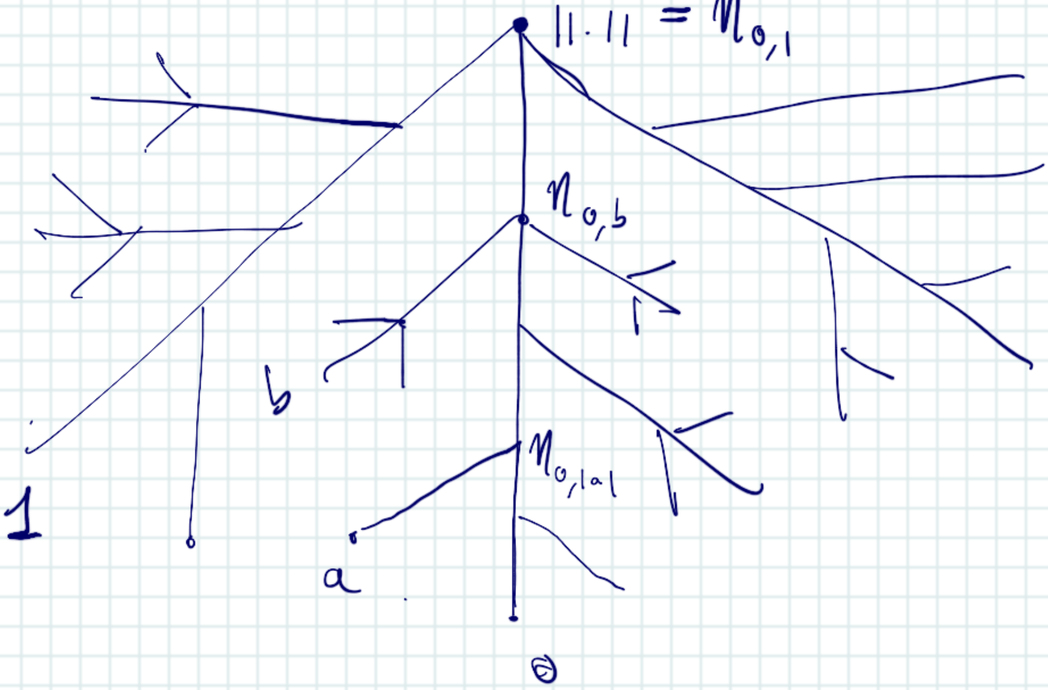
$a \in \mathcal{K}$
$1 \geq r > 0$ $\rightsquigarrow \eta_{a,r} : f \longmapsto \sup_{z \in B^+(a,r)} |f(z)|$

$\left(\text{e.g. } \eta_{0,r}(f) = \sum_{i=0}^{\infty} |a_i| r^i \right)$

So, as before we have homeomorphisms

$$\begin{aligned} [0,1] &\longrightarrow \mathcal{M}(k\{T\}) \\ r &\longmapsto \eta_{a,r} \end{aligned}$$

but now the intervals are <u>not</u> disjoint!



(if k not trivially valued)

<u>Theorem</u> (Berkovich)

If k is algebraically closed and spherically complete then all pts of $\mathcal{M}(k\{T\})$ are of the form $\eta_{a,r}$ $\exists\, a \in k,\ 0 \leq r \leq 1$

Lecture 2 + $\frac{1}{2}$ (more details on Banach rings and their spectra).

1) $\mathcal{M}(-)$ as a functor

$$\mathcal{M} : \text{BanRings} \longrightarrow \text{TopSpaces}$$

NB Morphisms in BanRings are bounded ring morphisms (hence $\mathbb{Z}$, $\|\cdot\|_\infty$ is initial object) /

If $\varphi : A \to B$ is a morphism of Banach rings,

$$\varphi^* : \mathcal{M}(B) \to \mathcal{M}(A)$$
$$x \longmapsto (f \mapsto x(\varphi(f)))$$

- well-defined OK
- continuous

Some issues:

- $\mathcal{M}$ is not fully faithful; not clear what BanRings^{op} should be geometrically.
  (it is not injective on objects)

- Another way to look at the topology on $\mathcal{M}(A) =: X$

$x \in X \rightsquigarrow |f(x)| := x(f)$ (idea: $f \in A$ are functions, $x \in X$ are points)

$\Rightarrow U \subset X$ open is a union of

$$\{x \in X \,/\, |f_i(x)| < p_i, \ |g_j(x)| > q_i, \ 1 \leq i \leq m, \ 1 \leq j \leq n\}$$

2) Norms on Tate algebras

k non-archimedean

$A = k\{r^{-1}T\} \ni f = \sum_{i=0}^{\infty} a_i T^i$, $\|f\|_1 = \sum |a_i| r^i < +\infty$.

$\rho(f) := \lim_{n\to\infty} \|f^n\|^{1/n}$ is non-archimedean and power multiplicati

Spectral radius

__Exercise__: • $\rho(f) = \max_i \{|a|_i r^i\}$ (hence it is a norm)

__Proposition__: $\mathcal{M}(k\{r^{-1}T\}, \|\cdot\|) \cong \mathcal{M}(k\{r^{-1}T\}, \rho\}$

__proof__. Consider the natural map $\xleftarrow{\phi}$.

- injective b/c $A \to A^\vee$ dense
- surjective b/c $\forall f \in A$ we have

$$|f(x)| = |f^n(x)|^{1/n} \le \|f\|^{1/n} \quad (\text{now let } n \to \infty)$$

so ϕ is continuous & bijective from compact to Hausdorff

Lecture 3 : Affinoid spaces and analytification

Correction 1 :

$$k\langle r^{-1}T\rangle = \{ f = \sum_i a_i T^i \,/\, \sum_i |a_i| r^i < +\infty \}$$

$$\cap \!\!\!\!\times$$

$$\xrightarrow{\text{as Banach algebra}} k\{r^{-1}T\} = \{ f = \sum_i a_i T^i \,/\, \lim_{i\to\infty} |a_i| r^i = 0 \}$$

this makes sense only for k non-archimedean

Correction 2 :

let $\mathbb{D}(1) = \mathbb{D}(0,1) := \mathcal{M}(k\{T\})$

$$\mathbb{D}(a,\rho) := \mathcal{M}(k\{\rho^{-1}(T-a)\}).$$

$$\begin{matrix}\rho<1\\ |a|<1\end{matrix} \Rightarrow \mathbb{D}(a,\rho)^+ \hookrightarrow \mathbb{D}(0,1)^+$$

$$\eta_{a,\rho}(f) := \sup_{x\in\mathbb{D}(a,\rho)^+} \{|f(x)|\} \qquad \left(\begin{matrix}\text{recall}\\ |f(x)| := x(f)\end{matrix}\right),$$

Quotient norm : given $(\mathcal{A}, \|\cdot\|)$ Banach ring and $I \lhd \mathcal{A}$, the quotient seminorm is

$$\begin{aligned} \|\cdot\|_q : \mathcal{A}/I &\longrightarrow \mathbb{R}_{\geq 0} \\ a &\longmapsto \inf\{\|A\| \,/\, A \equiv a \mod I\} \end{aligned}$$

If I closed then $\left(A/_I, \|\cdot\|_q\right)$ is a Banach ring

Affinoid algebras $\left(k \overset{\text{complete}}{\text{non-archimedean}}\right)$

Def. For every $\underline{r} = (r_1, -, r_n) \in (\mathbb{R}_{>0})^n$ let

$$T_n(\underline{r}) := k\{\underline{r}^{-1}\underline{T}\} := \left\{ f = \sum_{I \in \mathbb{Z}^n_{\geq 0}} a_I \underline{T}^I \;\Big/\; a_I \in k, \lim_{|I| \to \infty} |a_I| \underline{r}^I = 0 \right\}$$

Exercises: - show that $\|f\| = \max_I |a_I| r^I$ is a multiplicative norm and that $(k\{\underline{r}^{-1}\underline{T}\}, \|\cdot\|)$ is a Banach algebra

- what is this ring over $(k, |\cdot|_0)$?

FACTS: $T_n(\underline{r})$ is Noetherian, every ideal is closed.

Def. A Banach k-algebra $(A, \|\cdot\|_A)$ is affinoid if there is a surjective ring homomorphism

$$\varphi : k\{\underline{r}^{-1}\underline{T}\} \longrightarrow A$$

such that $\|\cdot\|_A \sim \|\cdot\|_q$ on $k\{\underline{r}^{-1}\underline{T}\}/_{\ker\varphi}$

It is strictly affinoid if we can take $\underline{r} = (1, -, 1)$.

A (strictly) k-affinoid space is a top. space of the form $\mathcal{M}(A)$ with A (strictly) k-affinoid

Examples:

- polydiscs : $\mathbb{D}(\underline{r}) := \mathcal{M}(k\{\underline{r}^{-1}\underline{T}\})$ $\left[\text{these are strict iff } r_i \in |k^\times| \ \forall i \ !\right]$
- polyannuli: pick $\underline{r} = (r_1, -, r_n)$; $\underline{s} = (s_1, -, s_n) \in \mathbb{R}^n_{>0}$ s.th. $s_i \leq r_i \quad \forall i$.

Set $\mathbb{A}(\underline{r}, \underline{s}) := \mathcal{M}\left(k\{\underline{r}^{-1}\underline{T}, \underline{s}\,\underline{U}\} \big/ (T_iU_i - 1)\right)$

Note: $\forall \ x \in \mathbb{A}(\underline{r}, s)$, we have

$$s_i \leq |T_i(x)| \leq r_i .$$

$$r_i^{-1} \leq |U_i(x)| \leq s_i^{-1}$$

Lemma (Noether normalization)

If $\mathcal{A}$ is strictly k-affinoid $\exists \ d \geq 0$ and a finite morphism $k\{T_1, -, T_d\} \hookrightarrow \mathcal{A}$.

Corollaries

- Let $\mathcal{A}$ be k-affinoid, $\mathfrak{m} \lhd \mathcal{A}$ maximal ideal. Then $\dim_k(\mathcal{A}/\mathfrak{m}) < \infty$.
- $|\cdot|_k$ extends uniquely to $\mathcal{A}/\mathfrak{m}$ ⇝ we have

$$\mathrm{Max}(\mathcal{A}) \hookrightarrow \mathcal{M}(\mathcal{A}).$$ [FACT: the image is dense]

(For A k-affinoid we also have:

- Noetherian
- all ideals closed)

The category k-Aff

Definition: Let k-aff be the full subcategory of Banach rings whose objects are k-affinoid algebras.

Then we set $k\text{-Aff} := (k\text{-aff})^{op}$

$\uparrow$ objects are $\mathcal{M}(A)$ with A k-affinoid.

Fiber products: in k-aff $A \times B$ is given by the completed tensor product $A \hat{\otimes}_k B$;

(exercise: which norm goes on the tensor product?)

"**structure sheaf**" $V \subset \mathcal{M}(A)$ is called affinoid domain if $\exists$ $A \xrightarrow{\varphi} A_V$ satisfying:

$\nwarrow$ k-affinoid

1 - $\varphi : \mathcal{M}(A_V) \longrightarrow \mathcal{M}(A)$ has V as image

2 - $\forall$ $A \to B$ $\rightsquigarrow$

s.th. $\psi(\mathcal{M}(B)) \subset V$

$$\begin{tikzcd}
\mathcal{M}(B) \arrow[r, "\psi"] \arrow[dr, "\exists !"'] & \mathcal{M}(A) \\
& \mathcal{M}(A_V) \arrow[u]
\end{tikzcd}$$

Fact (Tate acyclicity) $X = \mathcal{M}(\mathcal{A}) = \bigcup_i V_i$

with V_i affinoid domain $\forall i$.

$$\rightsquigarrow \quad 0 \longrightarrow \mathcal{A} \longrightarrow \prod_i \mathcal{A}_{V_i} \longrightarrow \prod_{i,j} \mathcal{A}_{V_i \cap V_j} \longrightarrow \cdots$$

$$(f_i) \longmapsto (f_i - f_j)$$

is exact.

Comments:

- if affinoid domains were the open sets in our topology then $V \longrightarrow \mathcal{A}_V$ would make a sheaf.
- one can extend the result to Berkovich-open sets

Global Berkovich spaces

(cf. étale cohomology paper* or Conrad notes §5)

$(X, \tau, \mathcal{A})$

net (specifies the affinoid domains)

atlas (how to associate algebras to elements of τ ?)

satisfying extra conditions. Good to know: one can glue affinoids along subschemes

* Etale cohomology for non-Archimedean analytic spaces, IHES, 1993

Lecture 4 - Analytification of schemes

Let X be a scheme of finite type over k.

Goal: define a Berkovich space X^{an}.

- $X = \mathbb{A}^n_k = \operatorname{Spec}(k[T_1, \dots, T_n])$

As a top. space

$$\mathbb{A}^{n,an}_k = \left\{ \begin{array}{l} \text{mult. seminorms} \\ \text{on } k[T_1, \dots, T_n] \end{array} \text{extending } |\cdot|_k \right\}$$

with the pointwise convergence topology.

It comes with a continuous map

$$\xi : \mathbb{A}^{n,an}_k \longrightarrow \mathbb{A}^n_k$$

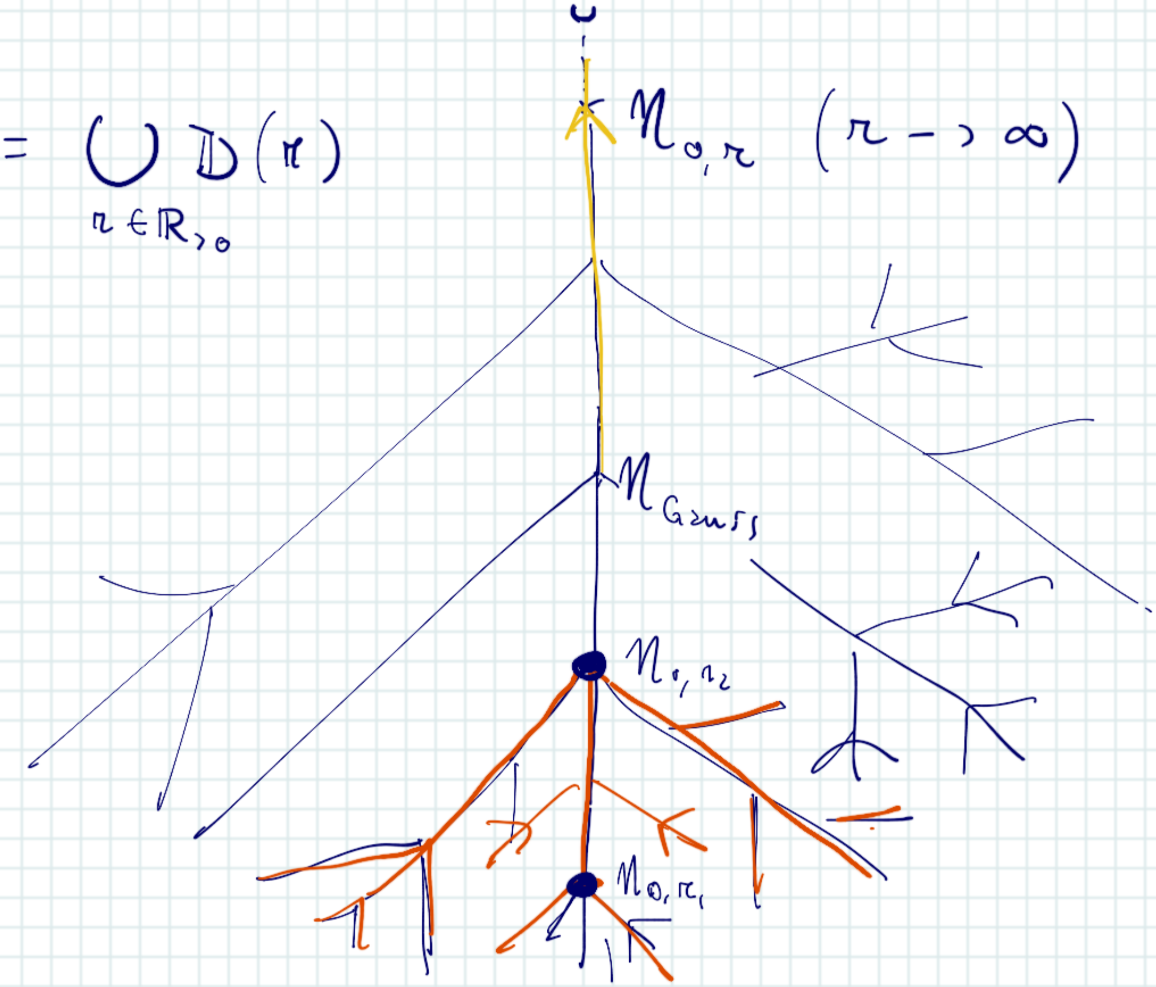
$$x \longmapsto \operatorname{Ker}(x)$$

and can be written as increasing union of affinoids:

$$\mathbb{A}^{n,an}_k = \bigcup_{\underline{r}} \mathbb{D}(\underline{r}) \qquad \mathcal{A}_{\mathbb{D}(\underline{r})} = k\{\underline{r}^{-1} \underline{T}\}.$$

Rmk. $\mathbb{A}^{n,an}_k$ is locally compact

$$\mathbb{A}^{1,an}_{k} = \bigcup_{r \in \mathbb{R}_{>0}} \mathbb{D}(r)$$



k alg. closed and spherically complete

$\rightsquigarrow$ $x \in \mathbb{A}^{1,an}_{k}$ can be of 3 kinds:

(1) $x = \eta_{a,0}$ $\exists$ $a \in k$

(2) $x = \eta_{a,r}$ $\exists$ $a \in k$, $r > 0$, $r \in |k^{\times}|$

(3) $x = \eta_{a,r}$ $\exists$ $a \in k$, $r > 0$, $r \notin |k^{\times}|$

- $X = \mathrm{Spec}(A)$, $A = k[T_1, \dots, T_n]/I$

$$X^{an} = \{ \text{mult. seminorms } A \longrightarrow \mathbb{R}_{\geq 0} \text{ extending } |\cdot|_k \}$$

$X^{an} \xhookrightarrow{\iota} \mathbb{A}_k^{n,an} \rightsquigarrow \iota^{-1}(\mathbb{D}(r))$ is either empty or affinoid domain of X^{an}.

$$\bigcup_r \left(\iota^{-1}(\mathbb{D}(r)) \right) = X^{an}.$$

- X general $\qquad X_{ij} \xrightarrow{\overset{\varphi}{\sim}} X_{ji}$

Then $X^{an} = \bigcup_i X_i^{an}$, $\quad X_{ij}^{an} \xrightarrow{\overset{\varphi^{an}}{\sim}} X_{ji}^{an}$.

There is a map

$$\xi : X^{an} \longrightarrow X \quad \text{obtained by glueing.}$$

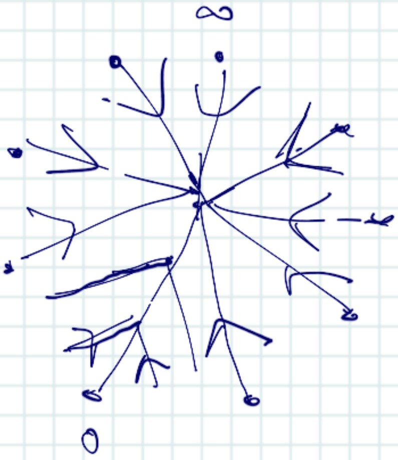
$$\xi^{-1}(x) = \left\{ \begin{matrix} \text{mult. norms on } k(x) \\ \text{extending } |\cdot|_k \end{matrix} \right\}$$

nice description, but only set-theoretic ↙

$$\boxed{X^{an} = \left\{ (x, |\cdot|) : x \in X \text{ and } \begin{matrix} |\cdot| : k(x) \to \mathbb{R}_{\geq 0} \\ \text{extending } |\cdot|_k \end{matrix} \right\}}$$

Example $\mathbb{P}^{1,an}_k$

$$\mathbb{P}^{1,an}_k = \overset{T_0}{\mathbb{A}^{1,an}_k} \sqcup \overset{T_1}{\mathbb{A}^{1,an}_k} \Big/ \sim \quad \left(\begin{array}{l} \text{over} \\ |T_0| \neq 0 \\ |T_1| \neq 0 \end{array} \rightsquigarrow \begin{array}{c} \text{apply the map} \\ T_0 \longmapsto T_1^{-1} \end{array} \right)$$



Theorem (Berkovich) [chapter 3 of Berkovich's book]

- X^{an} is locally compact and locally path-connected
- X connected $\iff$ X^{an} connected
- X separated $\iff$ X^{an} Hausdorff
- X proper $\iff$ X^{an} compact Hausdorff
- $\dim(X) = \dim(X^{an})$

↖ topological dimension (in the sense of normal spaces)

If k alg. closed, sph complete, $x \in \mathbb{P}^{1,an}_{k}$ then 3 cases:

(1) $\mathrm{Ker}(x)$ is a closed pt

(2) $\mathrm{Ker}(x) = (0)$ and $|\mathcal{H}(x)^\times| / |k^\times|$ is finite

(3) $\mathrm{Ker}(x) = (0)$ and $|\mathcal{H}(x)^\times| / |k^\times|$ is infinite

Note that:

(1) $\Longleftrightarrow$ $\mathbb{P}^{1,an} \setminus \{x\}$ is connected

(2) $\Longleftrightarrow$ $\mathbb{P}^{1,an} \setminus \{x\}$ has 2 connected components

(3) $\Longleftrightarrow$ $\mathbb{P}^{1,an} \setminus \{x\}$ has ∞ " "

Another example: elliptic curve (res. char $\neq 2$)

E/k elliptic curve $\rightsquigarrow$ $E^{an} = ?$

$y^2 = x(x-1)(x-\lambda)$ $\qquad$ $\pi^{an}: E^{an} \longrightarrow \mathbb{P}^{1,an}_{k}$ ramified at $\{0,1,\lambda,\infty\}$

Given a pt $\eta_{a,\rho} \in \mathbb{A}^{1,an}_{k}$, what's $(\pi^{an})^{-1}(\eta_{a,\rho})$?

$$\begin{tikzcd}
? & k(E) = k(x)[y] / y^2 - x(x-1)(x-\lambda) \arrow[l] \\
\mathcal{H}(\eta_{a,\rho}) \arrow[u] & k(x) \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

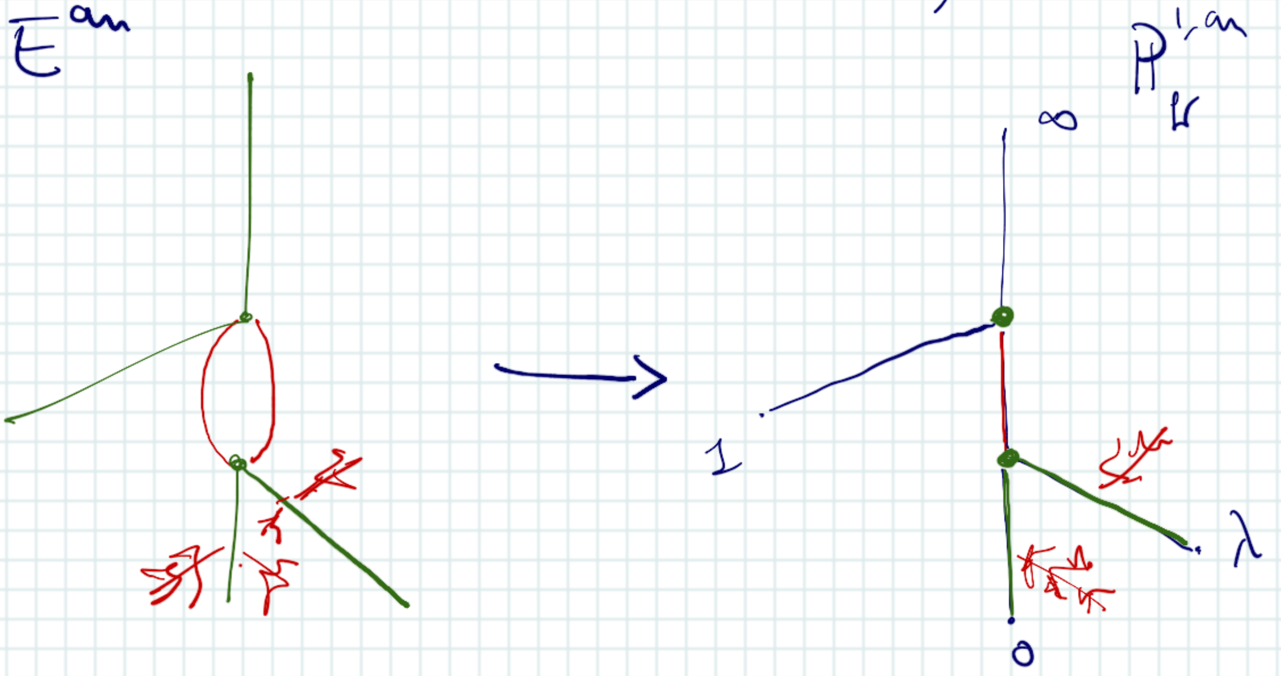
Case 1 : $0<|\lambda|<1$, $\rho>0$

$$\sqrt{x(x-1)(x-\lambda)}\begin{cases} \in \mathcal{H}(\eta_{a,\rho}) & a\neq 0,1,\lambda \\ \notin \mathcal{H}(\eta_{0,\rho}) & \rho\leq|\lambda| , \rho\geq 1 \\ \in \mathcal{H}(\eta_{0,\rho}) & |\lambda|<\rho<1 \\ \notin \mathcal{H}(\eta_{1,\rho}) & \forall \rho \end{cases}$$

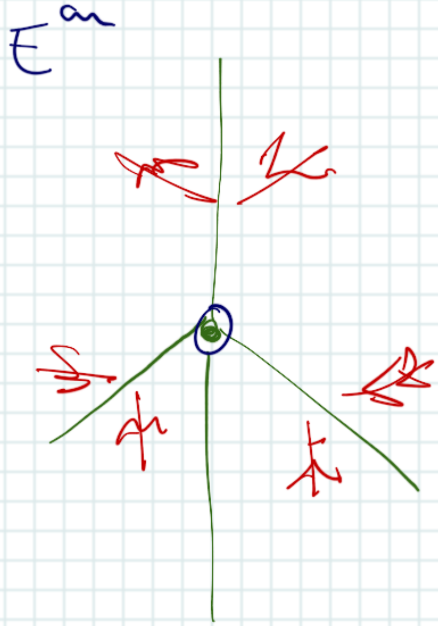
- $\sqrt{x-1} = i\sqrt{1-x} = i\left(1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\cdots\right)$
- $Z^2 - x(x-\lambda)$ has a solution iff $|x|>|\lambda|$

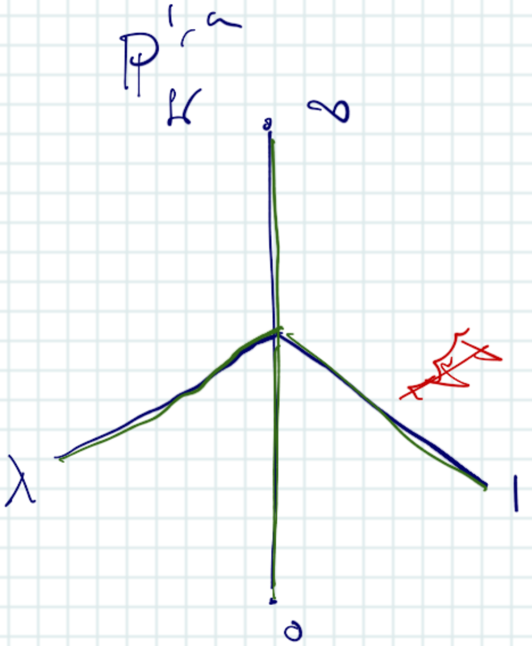
(reason in $\widetilde{\mathcal{H}(\eta_{0,\rho})}$ and use Hensel's lemma)

$\uparrow$ converges for $|x|<1$



Case 2 : $|\lambda| = 1$, $|\lambda - 1| = 1$





Case 1 is a Tate curve:

$$E^{an} \cong k^\times / q^{\mathbb{Z}}$$

Case 2 is an elliptic curve with good reduction

(Note: the result is true without the assumption res char $\neq 2$, but it can't be proved in the same way)

Lecture 5 · Berkovich curves

In the first part, we prove the following:

Theorem (Berkovich, Baker-Payne-Rabinoff)

Let k be a non-archimedean field, $C/_k$ smooth projective curve. Then there exists a finite graph Σ and a continuous retraction $C^{an} \longrightarrow \Sigma$

§ Elements of proof 1: models

$k^\circ = \{x : |x| \leq 1\}$ is a local ring

$\mathfrak{m} = \{x : |x| < 1\}$ its maximal ideal

$\tilde{k} = k^\circ/\mathfrak{m}$ a field $\left(\text{e.g. } k = \mathbb{C}((t)),\ k^\circ = \mathbb{C}[[t]],\ \tilde{k} = \mathbb{C} \right.$
$\left. k = \mathbb{Q}_p,\ k^\circ = \mathbb{Z}_p,\ \tilde{k} = \mathbb{F}_p \right)$

Let $C/_k$ be a smooth projective curve.

A model $\mathcal{C}$ of C is a flat, proper k°-curve such that $\mathcal{C} \underset{k^\circ}{\times} k = C$.

C is the generic fiber of C

$\mathcal{C}_{\tilde{k}} = \mathcal{C} \times_{k^\circ} \tilde{k}$ is a projective $\tilde{k}$ curve called special fiber of $\mathcal{C}$

$\mathcal{C}$ is called semi-stable if $\mathcal{C}_{\tilde{k}}$ is reduced and all singularities are double ordinary $\left(\overset{\forall}{P} \text{ with } \hat{\mathcal{O}}_{\mathcal{C}_{\tilde{k}},P} \cong \tilde{k}[[u,v]]/uv \right)$

Recalls

- stable $\Rightarrow$ semi-stable

- If C has semi-stable model, $g \geq 2$ then it has a unique stable model (= minimal s.s. model)
- There is a finite extension k'/k such that $C_{k'}$ has a semi-stable model

(this is a deep theorem by Deligne and Mumford. has applications to properness of $\overline{\mathcal{M}}_{g,n}$)

in particular, if k alg. closed, there is a semistable model.

§ Elements of proof 2 : the reduction map

Let $C_{/k}$ smooth projective curve

$\mathcal{C}_{/k^\circ}$ model of C

Want to build a map

$$\mathrm{red}_{\mathcal{C}} : C^{an} \longrightarrow \mathcal{C}_{\tilde{k}}$$

$x \in C^{an}$ ↓ defines a character $(A \to \mathcal{H}(x))$

⤳ gives a morphism

$$\begin{tikzcd}
\mathrm{Spec}(\mathcal{H}(x)) \arrow[r] \arrow[d, hook, "\circ" description] \arrow[dr] & C \arrow[d, hook, "\circ" description] \\
\mathrm{Spec}(\mathcal{H}(x)^\circ) \arrow[r, dashed, "\exists !"] & \mathcal{C} \\
\mathrm{red}_{\mathcal{C}}(x) : \mathrm{Spec}(\widetilde{\mathcal{H}(x)}) \arrow[u, hook, "\times" description] \arrow[r] & \mathcal{C}_{\tilde{k}} \arrow[u, hook, "\times" description]
\end{tikzcd}$$

$\mathcal{C}$ <u>proper</u> (use valuative criterion)

Proposition

(1) $\mathrm{red}_{\mathcal{C}}$ is anticontinuous: pre-images of opens are closed

(2) If ξ generic pt of irred comp of $\mathcal{C}_{\tilde{k}}$ then $\mathrm{red}^{-1}(\xi)$ is a single pt (assuming normality of the model)

Berkovich

(3) $P \in \mathcal{C}_{\tilde{k}}$ is smooth $\iff$ $\mathrm{red}^{-1}(P)$ is an open disc

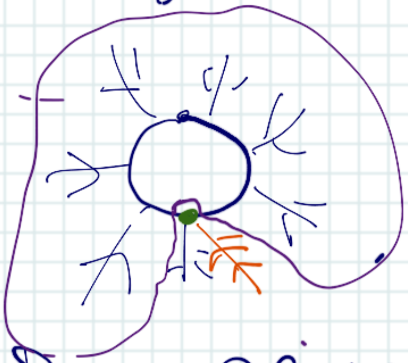
$P \in \mathcal{C}_{\tilde{k}}$ is double ordinary $\iff$ $\mathrm{red}^{-1}(P)$ is an open annulus

Bosch Lütkebohmert

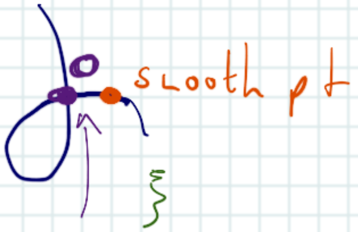
recall how do these look

Example $\mathcal{C}$ $y^2 = x(x+1)(x+t)$ over $\tilde{k}[[t]]$

C^{an} :



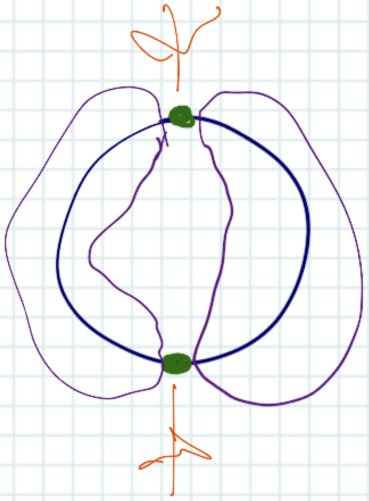
$\mathcal{C}_{\tilde{k}}$:



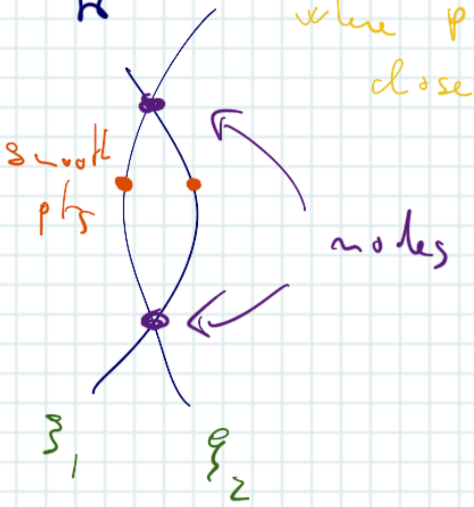
Note: the green node point is the only possibility for the boundary of $\mathrm{red}^{-1}(P)$ where P is a closed point

$\mathcal{D} := Bl_0 \mathcal{C}$

C^{an} :



$\mathcal{D}_{\tilde{k}}$



Elements of proof 3: vertex sets

Definition

A semistable vertex set of C^{an} is a finite set $V \subset C^{an}$ such that

$$C^{an} \setminus V = \left(\underset{\overset{\nearrow}{\text{infinite}}}{\coprod} B_\alpha \right) \coprod_{i=1}^{n} A_i$$

Theorem (Baker - Payne - Rabinoff)

There is a bijection

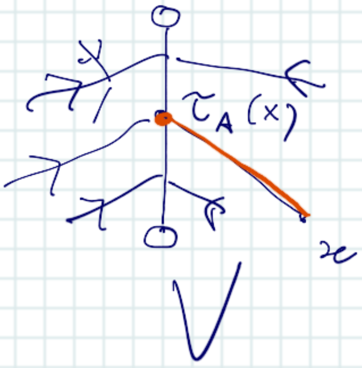
$$\left\{ \begin{matrix} \text{s.s. vertex} \\ \text{sets of } C^{an} \end{matrix} \right\} \longleftarrow \left\{ \begin{matrix} \text{s.s. models} \\ \text{of } C \end{matrix} \right\}$$

$$V_{\mathcal{C}} \longleftarrow\!\shortmid \ \mathcal{C}$$

s. th. $V_{\mathcal{C}} = \left\{ \mathrm{red}^{-1}(\xi) : \xi \text{ is gen. pt. of irred. comp. of } \mathcal{C}_{\tilde{k}} \right\}$

Proof of theorem

- For any open annulus $A \subset C^{an}$, define its skeleton $\Sigma_1(A)$



and a retraction $\tau_A : A \to \Sigma_1(A)$ defined in two steps:

$$\begin{tikzcd}
A \arrow[rr, "\tau_A"] \arrow[dr, "trop"'] & & \Sigma_1(A) \\
& \mathbb{R} \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

$trop : A \longrightarrow \mathbb{R}$
$x \longmapsto -\log|T(x)|$

$\sigma : \mathbb{R} \longrightarrow \Sigma_1(A)$
$r \longmapsto \eta_{0,r}$

It is clearly continuous

Properties:

- τ_A can be upgraded to a strong deformation retract (Berkovich's book - Proposition 4.1.6.)
- $x \in A$, $D \subset A \setminus \{\tau_A(x)\}$ connected component containing x. Then $\tau_A(x) = \partial(D) =$ limit pts of D not in D

- Let V be a semi-stable vertex set.

Let
$$\Sigma_1 = V \cup \left(\dot{\bigcup_{A \in \mathcal{A}_V}} \Sigma_1(A) \right)$$

Connected components of $C^{an} \setminus \Sigma_1$ are open discs.

- Define
$$\tau : C^{a} \longrightarrow \Sigma_1$$
$$x \longmapsto \begin{cases} x & \text{if } x \in \Sigma_1 \\ \partial(D) & \text{if } x \in C^{an} \setminus \Sigma_1 \underset{\text{conn comp}}{\supset} \overset{\overset{x}{\wedge}}{D} \end{cases}$$

- show that τ is a deformation retraction

↳ follows from: $x \in V$, U open neighborhood

$\Rightarrow \tau^{-1}(U)$ open ($\cong$ complement of finite union of closed discs in a suitable curve)

$\square$

Lecture 6: Curves and beyond

Today k is non-archimedean, non-trivially valued.
(complete)

The non-algebraically closed case

Recall: $X^{an}/_{k} \rightsquigarrow X^{an} \cong X^{an} \times_{k} \widehat{\bar{k}} \Big/ \mathrm{Gal}(\bar{k}|k)$

e.g. $x \in \mathbb{P}^{1,an}_{\mathbb{Q}_p}$ is a Galois orbit of $y \in \mathbb{P}^{1,an}_{\mathbb{C}_p}$,

$p = 3$ $\quad x = [\eta_{\sqrt{-1}, \frac{1}{2}}]$

$\mathcal{H}(x) = \widehat{\mathbb{Q}_p[\sqrt{-1}](X)}$,

Problem: the action of $\mathrm{Gal}(\bar{k}|k)$ is not always well understood (especially when $\mathrm{char}(\tilde{k}) = p > 0$)

Types of pts: to generalize Berkovich classification, let us introduce some invariants.

$$x \in X^{an} \rightsquigarrow s(x) := \mathrm{trdeg}_{\tilde{k}} \widetilde{\mathcal{H}(x)}$$

$$t(x) := \dim_{\mathbb{Q}} \left(|\mathcal{H}(x)^{\times}| \big/ |k^{\times}| \otimes \mathbb{Q} \right)$$

Let $\xi = \ker(x) \in X$. Then Abhyankar's inequality gives:

[$\kappa(\xi)$ its residue field] $\quad s(x) + t(x) \leq \mathrm{trdeg}_{k} \kappa(\xi)$

TYPE	$\mathbb{P}^{1,an}_{\hat{k}}$	C^{an}/k
(1)	$\eta_{a,0} \rightsquigarrow \mathcal{H}(\eta_{a,0}) = k$	$\mathcal{H}(x) \subset \hat{k}$
(2)*	$\eta_{a,\rho}$ $\rho \in \lvert k^\times \rvert \rightsquigarrow s(\eta_{a,\rho}) = 1$	$s(x) = 1$
(3)	$\eta_{a,\rho}$ $\rho \notin \lvert k^\times \rvert \rightsquigarrow t(\eta_{a,\rho}) = 1$	$t(x) = 1$
(4)	$\mathcal{E} = \{D(a_i,\rho_i)\}$ with $\bigcap D(a_i,\rho_i) = \emptyset$ $D(a_{i+1},\rho_{i+1}) \subset D(a_i,\rho_i)$ $\forall i$ Let $\eta_{\mathcal{E}} = \inf_i(\eta_{a_i,\rho_i})$	$\ker(x) = 0$; $s(x) = 0$, $t(x) = 0$

* exercise: show that, in case (2)

$$\widetilde{\mathcal{H}(\eta_{a,\rho})} \cong \tilde{k}(t).$$

If $x \in C^{an}$ is of type (2) then $\widetilde{\mathcal{H}(x)}$ is the function field of a projective curve. We call genus of x, $g(x)$, the genus of such a curve.

Since $t(x) = 0$, then $|\mathcal{H}(x)^\times| / |k^\times|$ is a finite group. We call multiplicity of x, $m(x)$ its order

triangulations

$$C^{an}_{\widehat{\overline{H}}} \xrightarrow{\pi} C^{an}$$

An open $U \subset C^{an}$ is called

- virtual disc if $\exists$ an open disc $D \subset C^{an}_{\widehat{\overline{H}}}$ s.th. $\pi(D) = U$
- virtual annulus if $\exists$ an open annulus, $A \subset C^{an}_{\widehat{\overline{H}}}$ s.th. $\pi(A) = U$.

A triangulation of C^{an} is a finite set of type 2 pts $V \subset C^{an}$ such that

$$C^{an} \setminus V = \coprod_{\alpha} D_\alpha \coprod_{i=1}^{n} A_i$$

with D_α virtual discs and A_i virtual annuli.

Remarks

- semi-stable reduction theorem $\Longrightarrow$ every curve admits a triangulation.

- If $x \in C^{an}$ of type 2 is such that $g(x) > 1$ then $x \in V$ for every triangulation V

Theorem (Ducros)

If $g(C) \geq 2$ then there exists a minimal triangulation V_{min-tr} (i.e. for every triangulation V we have $V_{min-tr} \subset V$)

Theorem (Fantini - T.) Let K be discretely valued. Let L/K be the minimal extension such that C_L has a semi-stable model. Then

$$d(C) := \mathrm{lcm}\{ m(x) : x \in V_{min-tr} \} \mid [L:K]$$

Moreover, if L/K is tame then we have $d(C) = [L:K]$,

$V_{min-snc}$ is a triangulation, and $V_{min-tr} \subset V_{min-snc}$ is its principal subset.

where $V_{min-snc}$ is the vertex set corresponding to the minimal regular snc model and principal was

either · genus ≥ 1

or · connected to at least 3 other points in the dual graph

Remarks: the result above

- tamely ramified case ⇝ reproves results of T. Saito, L. Halle
- wildly ramified case ⇝ classification of pathologies

⇝ inspires further results in the case of potential multiplicative reduction (cf. Obus - T.)

Lecture 7 · Berkovich varieties and thin skeletons

(Nicaise, Mustaţa, Xu, Kollár, Thuillier, deFernex, Mazzon, ...)

[inspired by Kontsevich-Soibelman]

Suppose $X/_{k}$ smooth projective variety, k discretely valued,

Def. A regular model $\mathcal{X}$ of X is sncd if its special fiber $\mathcal{X}_{\tilde{k}}$ is an effective Cartier divisor with strict normal crossings:

$\dim \mathcal{X} = \dim(X) + 1$

[i.e. $\forall\, p \in \mathcal{X}_{\tilde{k}}$ $\exists\, T_1, \ldots, T_{\textcircled{d}} \in m_p \subset \mathcal{O}_{\mathcal{X},p}$ a regular system of parameters s.th. $\mathcal{X}_{\tilde{k}}$ is cut out by $T_1, \ldots, T_r$ in $\mathcal{O}_{\mathcal{X},p}$ $\leftarrow$ this means that $\pi = u \cdot \prod_{i=1}^{r} T_i^{N_i}$ in $\mathcal{O}_{\mathcal{X},p}$ with $u \in \mathcal{O}_{\mathcal{X},p}^{\times}$, $N_i \in \mathbb{N}$.

(In the case of curves this is equivalent to

$$\widehat{\mathcal{O}}_{\mathcal{X}_{\tilde{k}},p} \cong k[\![T_1, T_2]\!] \big/ T_1^{N_1} T_2^{N_2}$$
)

Def:

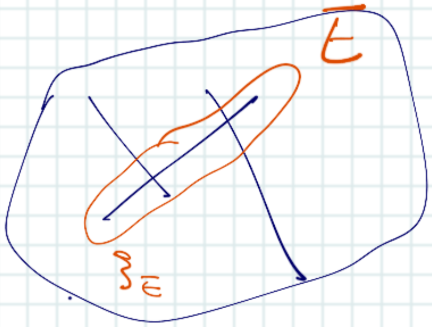
$$X^{bir} = \{ (\xi, |\cdot|) : \xi \text{ gen pt of } X \}$$

If X curve $\Rightarrow$ $X^{bir} = \{$ pts of type (2), (3) or (4) $\}$

Divisorial points

$\mathcal{X}$ normal model of X

$$\mathrm{red}_{\mathcal{X}} : X^{an} \longrightarrow \mathcal{X}_{\tilde{k}}$$



E irreducible component of $\mathcal{X}_{\tilde{k}}$, ξ_E its generic pt.

$$\mathrm{red}^{-1}(\xi_E) = \{\eta_E\}$$

Explicitly: $-\log(\eta_E) : k(X) \longrightarrow \mathbb{Z}$

$$f \longmapsto \frac{1}{N}\, \mathrm{ord}_E(f)$$

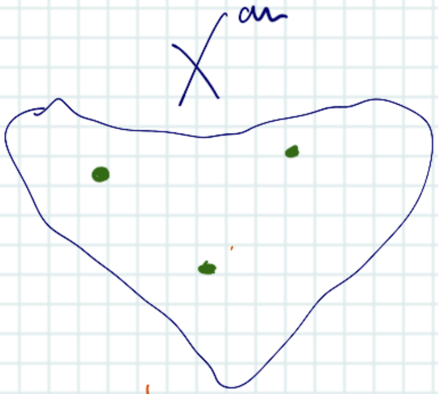
$$\left(\exists!\ N \text{ s.th. } \eta_E|_{k} = v_{k} \right)$$

$x \in X^{an}$ is **DIVISORIAL** if $\exists\, (\mathcal{X}, E)$ as above

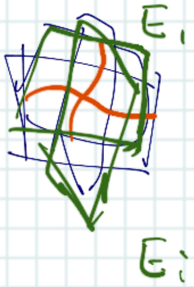
with $x = \eta_E$

Illustration:

$\mathcal{X}$



$$\mathcal{X}_{\tilde{k}} = \sum_{i=1}^{r} N_i E_i$$



Monomial points

Assume $\bigcap_{i=1}^{r} E_i \neq \emptyset$.

Let $\underline{\alpha} = (\alpha_1, -, \alpha_r) \in \mathbb{R}^{r}_{\geqslant 0}$ s.th. $\sum_{i=1}^{r} \alpha_i N_i = 1$.

Choose ξ gen. pt of an irreducible component of $\bigcap E_i$.

Proposition (Mustață – Nicaise)

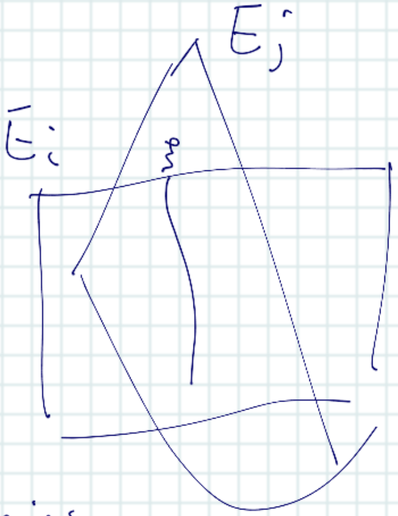
$\exists !$ minimal seminorm

$$|\cdot|_{\xi,\alpha} : \mathcal{O}_{\mathcal{X},\xi} \longrightarrow \mathbb{R}_{\geqslant 0}$$

s.th. $|T_i| = \alpha_i \quad \forall\, i \in \{1, -, r\}$

($T_i = 0$ is a local equation for E_i at ξ.)

Sketch of construction



$T_1, \ldots, T_r$ & $\mathcal{O}_{\mathcal{X},\xi}$ regular system of parameters

write $f \in \mathcal{O}_{\mathcal{X},\xi}$ as power series

$$f = \sum_{\beta \in \mathbb{N}^r} c_\beta T^\beta$$ and note that

$$v_{\xi,\alpha} : \mathcal{O}_{\mathcal{X},\xi} \longrightarrow \mathbb{R}_{\geq 0}$$
$$f \longmapsto \min\{\alpha \cdot \beta \,/\, \beta \in \mathbb{N}^r, c_\beta \neq 0\}$$

is a valuation (not depending on any choice made).

Extends uniquely to $K(X) = \mathrm{Frac}(\mathcal{O}_{\mathcal{X},\xi})$ and is compatible with v_K.

Hence $|\cdot|_K = \varepsilon^{v_K} \rightsquigarrow |\cdot|_\alpha = \varepsilon^{v_\alpha}$ is a point of X^{an}

Def. $x \in X^{an}$ is **monomial point** with respect to $\mathcal{X}$ if

$$\exists\ \xi, \alpha \text{ s.th. } x = |\cdot|_{\xi,\alpha}.$$

Remark about $X^{an} \supset X^{bir} \supset X^{mon} \supset X^{div}$

The skeleton $Sk_{\mathcal{X}}$

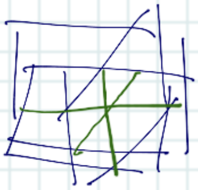
Definition: $Sk_{\mathcal{X}} = \{x \in X^{an} : x \text{ is monomial wrt } \mathcal{X}\}$

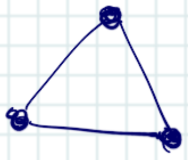
Properties:
- $Sk_{\mathcal{X}}$ is a simplicial complex
- $Sk_{\mathcal{X}} \simeq \Delta(\mathcal{X}_{\tilde{k}})$ dual complex of $\mathcal{X}_{\tilde{k}}$
- If $f \in K(X)^\times$ then

$$\begin{aligned} Sk_{\mathcal{X}} &\longrightarrow \mathbb{R} \\ x &\longmapsto \log|f(x)| \end{aligned}$$

is continuous & piecewise affine.

(Berkovich Thuillier) - $\exists$ a strong deformation retract of X^{an} over $Sk_{\mathcal{X}}$ for every sncd model $\mathcal{X}$ of X.

Example:
- X curve, $\mathcal{X}$ stable model $\Rightarrow Sk_{\mathcal{X}} = \Sigma(X)$
- X surface, $\mathcal{X}_{\tilde{k}} =$  (union of coordinate hyperplanes)

$\Rightarrow \quad SH_{\mathcal{X}} =$ 

Recall (dual complex)

$\mathcal{X}_{\tilde{k}} = \sum_{i\in I} N_i E_i$, $J \subset I \Rightarrow E_J := \bigcap_{j\in J} E_j$

$\mathcal{X}_{\tilde{k}} \rightsquigarrow \Delta(\mathcal{X}_{\tilde{k}})$ is defined as the simplicial complex whose ...

... d-simplices are connected components of E_J with $|J| = d+1$

glued along their faces in such a way that τ, τ' simplices,

connected components

$$\tau \subset \tau' \iff C_{\tau'} \subset C_{\tau}$$

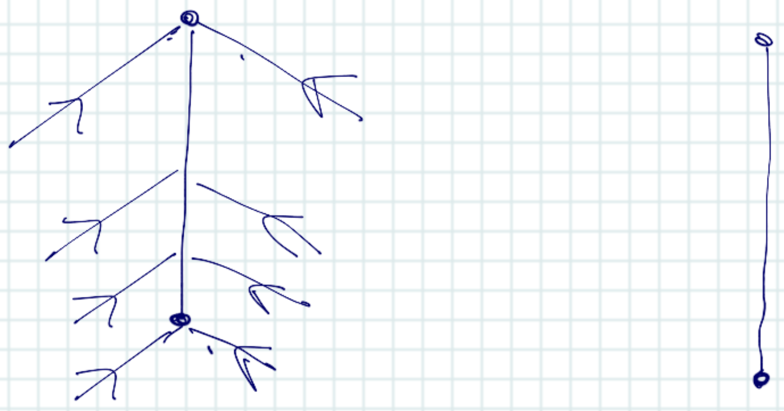
(so, vertices are irreducible components, 1-simplices represent the intersection of two components, etc.)

Lecture 8 · Skeletons as tropicalizations (k complete non-archimedean)

Motivation: skeletons of annuli

$$A(r,s) := \{ x \in \mathbb{A}^{1,an}_k : s \leq |T(x)| \leq r \}$$

$$\begin{tikzcd} A \arrow[r] & \Sigma(A) \end{tikzcd}$$



$$[-\log|r|, -\log|s|]$$

$$x \xmapsto{\mathrm{trop}_A} -\log|T(x)|$$

Q. Can we globalize this process to get skeletons of curves?

$$\begin{tikzcd}
\mathbb{T}^n_K := \operatorname{Spec} K[T_1^{\pm 1}, \dots, T_n^{\pm 1}] \arrow[d, squiggly] & \\
\mathbb{T}^{n,an}_K \arrow[r, "trop"] & \mathbb{R}^n
\end{tikzcd}$$

$$\longmapsto \left(-\log|T_1(x)|, \dots, -\log|T_n(x)|\right)$$

Let $\varphi : X \hookrightarrow \mathbb{T}^n_K$ be a closed immersion

$$\mathrm{Trop}_\varphi(X) := \mathrm{trop} \circ \varphi^{an}(X^{an})$$

Remark. trop is continuous $\Rightarrow$ trop_φ is continuous

Theorem (Bieri-Groves)

Let X be of pure dimension d.

For every φ, $\mathrm{Trop}_\varphi(X)$ is a finite union of d-dimensional, Γ-rational polyhedra

$$\left(\Gamma = |K^\times|\right)$$

Example: if k trivially valued, $\mathrm{trop}(X)$ is a finite union of rational polyhedral cones.

Properties

① • X connected $\Rightarrow$ $\mathrm{Trop}_\varphi(X)$ connected

(recall: X connected $\Rightarrow$ X^{an} connected!)

② • $k'|k$ complete extension of valued fields

$X_{k'} \xhookrightarrow{\varphi'} \mathbb{T}^n_{k'}$ Then

$$\begin{tikzcd}
\rightsquigarrow & X_{k'} \arrow[d] \arrow[dr, "\mathrm{Trop}_{\varphi'}"] & \\
& X_k \arrow[r, "\mathrm{Trop}_\varphi"] & \mathbb{R}^n
\end{tikzcd}$$

commutes

(b/c it commutes analytically and basechange is surjective on analytifications)

③ • $$\mathrm{Trop}_\varphi(X) = \overline{\left\{ (-\log|x_1|, \dots, -\log|x_n|) : \underset{(x_1, \dots, x_n)}{\underset{\shortparallel}{x}} \in X(\bar{k}) \hookrightarrow \mathbb{T}^n(\bar{k}) \right\}}$$

(by density of pts of type (1))

Initial degenerations & tropical multiplicity

$$X \hookrightarrow \mathbb{T}^{n,\mathrm{an}}_{K}$$

$K'|K$ valued field extension, $t \in \mathbb{T}^{n}_{K}(K')$

$$t^{-1}\cdot X_{K'} \hookrightarrow \mathbb{T}^{n}_{K'} \hookrightarrow \mathbb{T}^{n}_{(K')^{\circ}}$$

$\mathcal{X}^{t} := \overline{t^{-1}\cdot X_{K'}}$ is a $(K')^{0}$ scheme

Definition

The initial degeneration of X at t is

$$in_{t}(X) := (\mathcal{X}^{t})_{\tilde{K}} \hookrightarrow \mathbb{T}^{n}_{\tilde{K}}$$

Proposition: if $t' \in \mathbb{T}(K')$ $t'' \in \mathbb{T}(K'')$ are such that

$trop(t') = trop(t'') =: w$ then $\exists$ $\ell|K'$, $\ell|K''$ with

$$in_{t'}(X)_{\tilde{\ell}} = \widetilde{\left(\frac{t}{t'}\right)}\, in_{t''}(X)_{\tilde{\ell}}\,.$$

$\rightsquigarrow$ "up to translation, $in_{t}(X)$ depends only on $trop(t)$".

Example Let C plane curve of equation

$$y^2 = x^3 + x^2 + t^4 \quad \text{in} \quad \mathbb{G}^2_{m,\,\mathbb{C}((t))}$$

$w = (0,0) \rightsquigarrow \mathrm{in}_w(C) := \{(\tilde{x},\tilde{y}) \in \mathbb{G}^2_{m,\mathbb{C}} : \tilde{y}^2 = \tilde{x}^3 + \tilde{x}^2\}$

$W = (1,0)$

$$y^2 = t^3x^3 + t^2x^2 + t^4 \rightsquigarrow \mathrm{in}_w(C) := \{ -, \tilde{y}^2 = 0\}$$

$$\rightsquigarrow \mathrm{in}_w(C) = \emptyset$$

$w = (1,1)$

$$t^2y^2 = t^3x^3 + t^2x^2 + t^4$$

$$\Updownarrow$$

$$y^2 = tx^3 + x^2 + t^2$$

$$\rightsquigarrow \mathrm{in}_w(C) = \{ \qquad\qquad : \tilde{y}^2 = \tilde{x}^2\}$$ reducible

(2 conn comp)

Theorem (Kapranov theorem, Speyer-Sturmfels, Draisma, Payne)

$$\varphi : X \hookrightarrow \mathbb{T}^n_K$$

$$\mathrm{Trop}_\varphi(X) = \{ \omega \in \mathbb{R}^n \,/\, in_\omega(X) \neq \emptyset \}$$

Idea of proof

Fix $\omega \in \mathbb{R}^n$. WTS: $\omega \in \mathrm{Trop}_\varphi(X) \Longleftrightarrow in_\omega(X) \neq 0$

base change $\rightsquigarrow$ assume K non-trivially valued, algebraically closed

$\rightsquigarrow$ assume $\omega = \mathrm{trop}(\underline{1}) = \underline{0}$ (after translation)

$$\mathcal{X} := \mathcal{X}^{\underline{1}}$$

$$\begin{tikzcd} \mathcal{X}_\eta \arrow[r, "\mathrm{red}"] \arrow[d, equal] & \mathcal{X}_{\tilde{K}} \\ \mathrm{trop}^{-1}(0) \cap X^{an} & \end{tikzcd}$$

But red is surjective $\Rightarrow$ $\mathcal{X}_{\tilde{K}} = \emptyset \Longleftrightarrow \mathrm{trop}^{-1}(0) \cap X^{an} = \emptyset$

$\Longleftrightarrow 0 \notin \mathrm{trop}(X^{an})$.

Tropicalization of toric varieties (after Payne & Kajiwara)

$N \cong \mathbb{Z}^n$ lattice, $M = \mathrm{Hom}(N, \mathbb{Z})$

Δ fan in $N_{\mathbb{R}}$ $\rightsquigarrow$ Y_Δ corresponding toric variety

(collection of affine cones + ...)

$\cup$

$\mathbb{T}^n$ dense, acting on Y_Δ

Construction of $\mathrm{Trop}(Y_\Delta)$:

Let $\sigma \in \Delta$, $N(\sigma) = N_{\mathbb{R}} / \mathrm{Span}(\sigma)$

$$\underset{\text{set}}{\overset{\text{as}}{\rightsquigarrow}} \quad \mathrm{Trop}(Y) = \coprod_{\sigma \in \Delta} N(\sigma)$$

$$\begin{tikzcd} Y(K) \arrow[r] \arrow[rr, bend right, "\mathrm{Trop}"'] & Y^{an} \arrow[r, "\mathrm{trop}"] & \mathrm{Trop}(Y) \end{tikzcd}$$

obtained by glueing

$T_\sigma \to T_\sigma^{an} \longrightarrow N(\sigma)$.

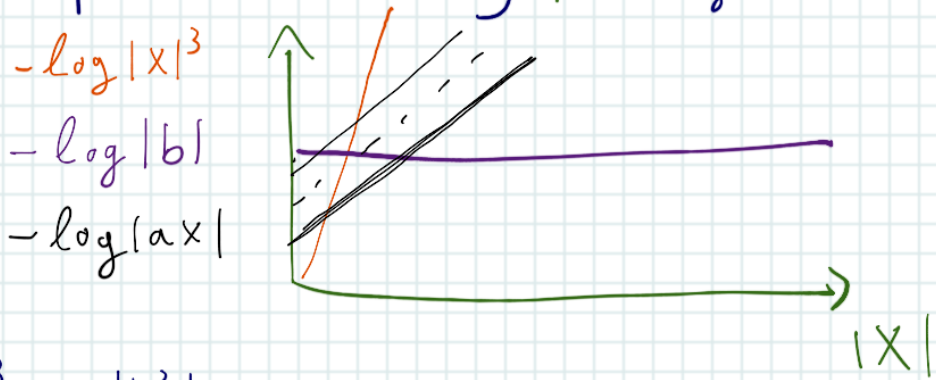
[$x \in Y^{an} \Rightarrow x \in T_\sigma^{an}$ $\exists\, \sigma \in \Delta$ (T_σ = quotient torus acting simply transitively on σ)]

Lecture 9: tropical curves and their moduli

Recall: Weierstrass example

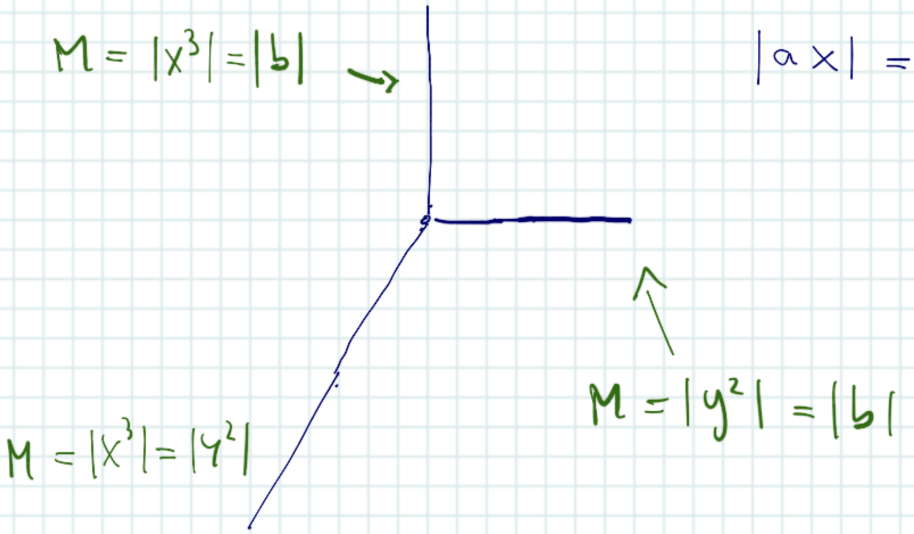
$a, b \in K^\circ$ $\quad y^2 = x^3 + ax + b \quad C \overset{\varphi}{\hookrightarrow} \mathbb{T}_K^2$

Compute tropicalization by looking at $in_w(C)$:



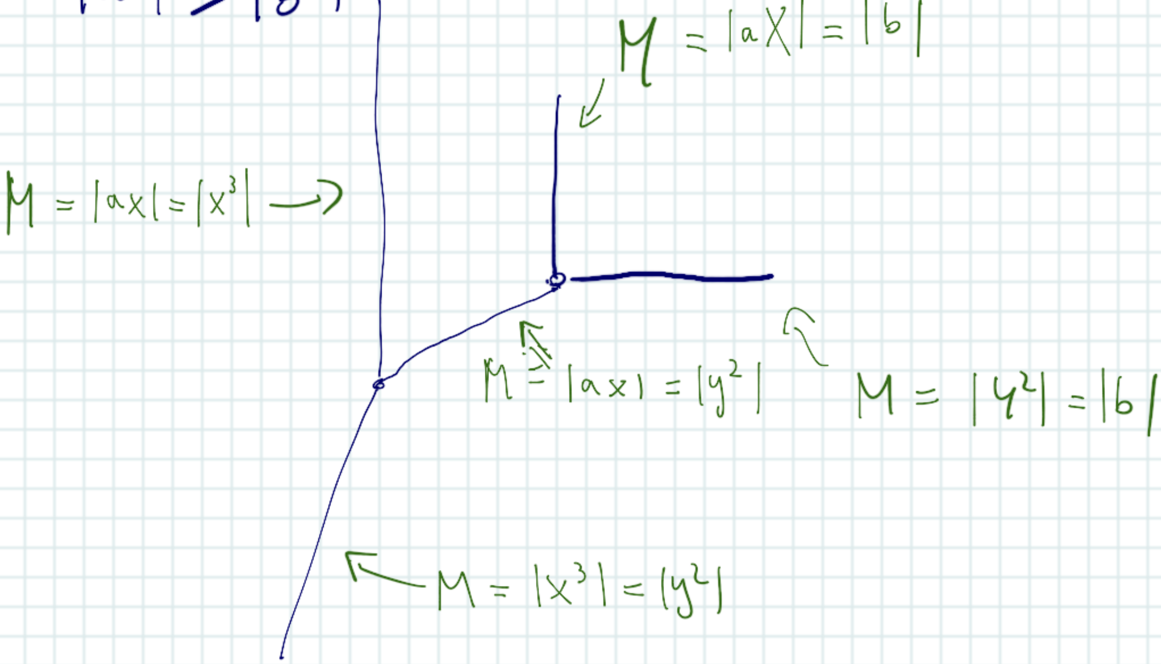
Case 1 $|a^3| \le |b^2|$

$|ax| = |x^3| \iff |a| = |x^2|$



Case 2 : $|a^3| > |b^2|$

$\mathrm{Trop}_\varphi(C)$:



Remark : if the Weierstrass equation has multiplicative reduction, the skeleton is a circle, something that the tropicalization doesn't see.

So, this tropicalization is not "faithful".

k alg. closed, non-trivially valued

$C/_k$ smooth curve.

Q: $\exists\ \varphi$ s.th. $\mathrm{trop}_\varphi(C^{an}) \supset \Sigma_1(C)$?

[Baker-Payne-Rabinoff:]

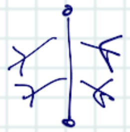
define a metric on C^{bin} (set of pts of type (2), (3), (4))

Idea: $C^{\mathrm{bin}} = \varinjlim_{V} \Sigma(C,V)$

semistable vertex sets

$\Rightarrow$ $x, y \in C^{\mathrm{bin}}$ lie on some $\Sigma(C,V)$.

Facts:

- shortest path does not depend on V
- edges of $\Sigma(C,V)$ can be metrized  $\simeq A(1,s)$

$$d(\eta_{0,a}, \eta_{0,b}) = \left| \log|a| - \log|b| \right|$$

$\Rightarrow$ d extends to a metric on C^{bin}.

On the other hand, $\mathrm{trop}_\varphi(C)$ can be metrized.

$d(x,y) :=$ lattice length of shortest path between x and y.

Theorem (BPR)

Let $\vec{\Sigma}_1 \subset C^{an}$ be a finite subgraph.

Then there is $\varphi : C \hookrightarrow \underset{\overset{\cup}{\mathbb{T}}}{Y}$ with Y toric

such that $\varphi|_{C \cap \varphi^{-1}(\mathbb{T})} : C \cap \varphi^{-1}(\mathbb{T}) \overset{\varphi_0}{\hookrightarrow} \mathbb{T}$ has

a faithful tropicalization (i.e. $\mathrm{trop}_{\varphi_0}$ induces an isometry $\vec{\Sigma}_1 \xrightarrow{\sim} \mathrm{trop}_{\varphi_0}(\vec{\Sigma}_1)$).

Corollary Let $g(C) \geq 2$.

The minimal skeleton $\overset{\vec{\Sigma}_1(C, V_{st})}{\overset{\shortparallel:}{\vec{\Sigma}_1(C)}}$ of C^{an} can be represented isometrically by a suitable tropicalization

(in some sense a **minimal meaningful** tropicalization)

Definition

$\vec{\Sigma}_1(C)$ is the abstract tropicalization of C.

(**mention** that there is a version of **minimal** skeleton for curves with marked points & everything carries over)

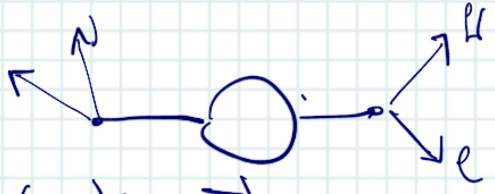
The space $M_{g,n}^{trop}$ (Brannetti _ Melo _ Viviani, Caporaso)

Lots of tropicalizations for the same curve

⇝ abstract tropicalizations are more suited for moduli

Abstract (n-marked) tropical curve Γ is the datum of (G, ℓ, g, m)

- G finite graph ($E(G)$ edges; $V(G)$ vertices)
- $\ell : E(G) \longrightarrow \mathbb{R}_{>0}$ length
- $g : V(G) \longrightarrow \mathbb{N}$ function
- $m : \{1, \dots, n\} \longrightarrow V(G)$ marking

(represented as 

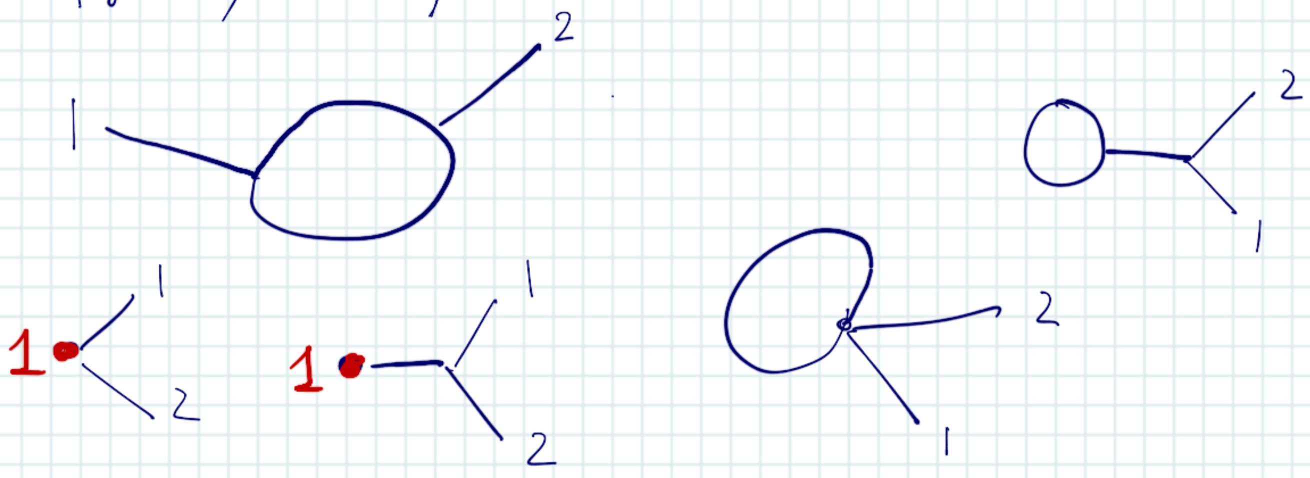
)

Genus: $g(\Gamma) = h_1(G) + \sum_{v \in V(G)} g(v)$.

Combinatorial type: (G, g, m)

Γ is stable if $\forall\, v \in V(G)$ $\quad 2g(v) - 2 + \mathrm{VALENCE}(v) + |m^{-1}(v)| > 0$

Ex. $(g=1,\ n=2)$



Fixed combinatorial type

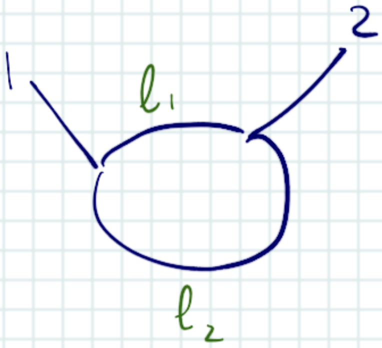
Fix g, n and a combinatorial type (G, g, m)

Build a parameter space:

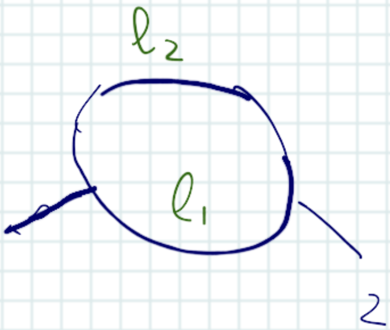
1. Specify edge lengths $\mathbb{R}_{>0}^{|E(G)|} \Big/ \mathrm{Aut}(G, g^{(\cdot)}, m^{(\cdot)})$

2. Quotient out by symmetries

e.g.



$\cong$

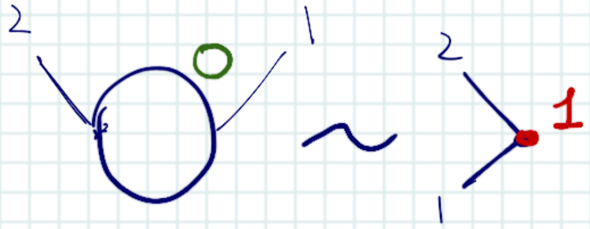


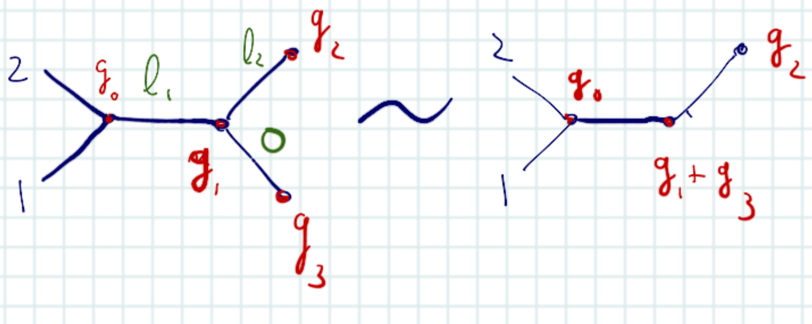
$$\overline{C(G,g,m)} = \mathbb{R}_{\geq 0}^{E(G)} \Big/ \mathrm{Aut}(G,g,m)$$

Definition

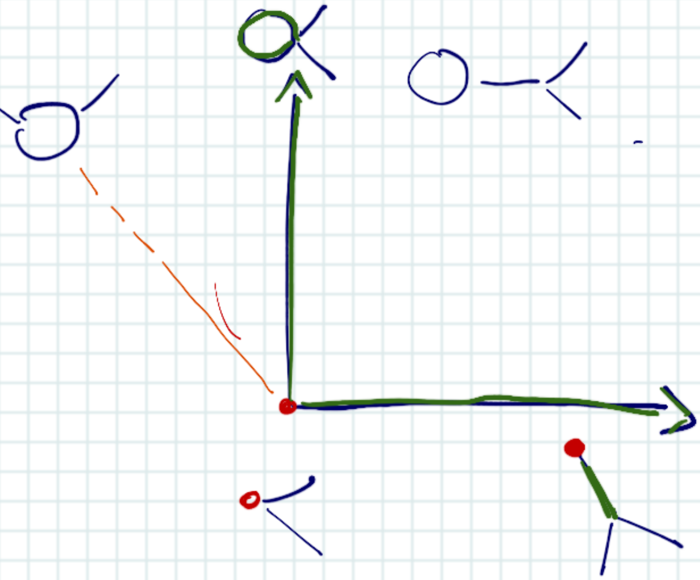
$$M^{trop}_{g,n} := \coprod_{\substack{\text{comb} \\ \text{types}}} \overline{C(G,g,m)} \Big/ \sim$$

where $\sim$ is generated by contraction of length 0 edges.

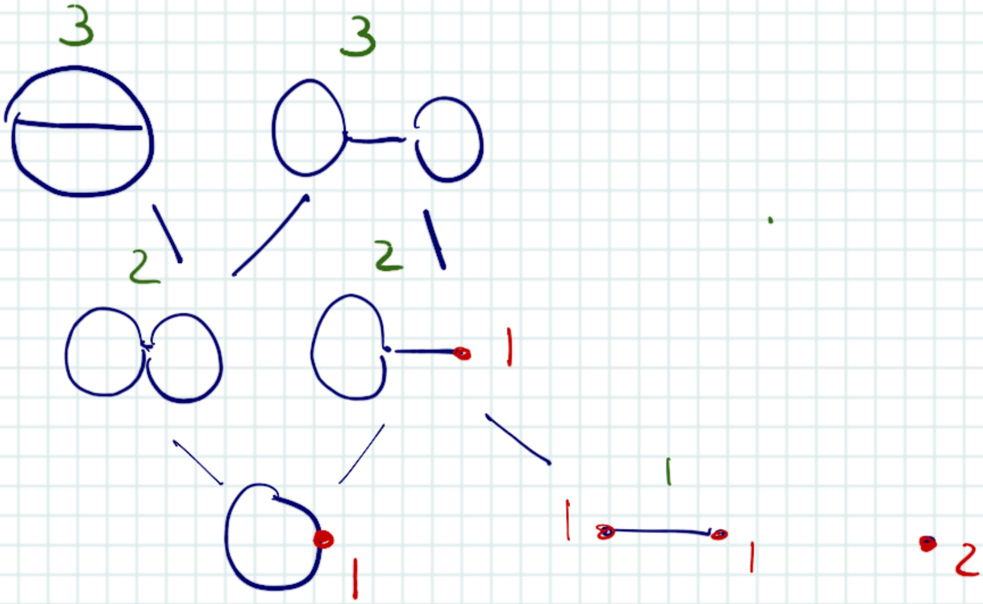


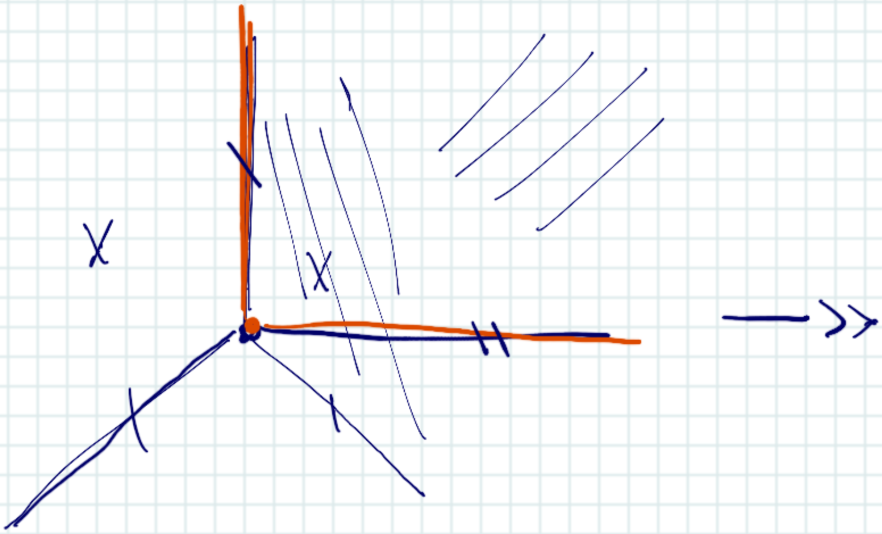


$M^{trop}_{1,2}$



M_2^{trop}





Lecture 10. $M_{g,n}^{trop}$ Vs $\mathcal{M}_{g,n}$

Recall: $M_{g,n}^{trop} = \coprod_{\substack{\text{comb} \\ \text{types}}} \overline{C(G,g,m)} \Big/ \sim$

Properties: $M_{g,n}^{trop}$ is a generalized cone complex of dimension $3g-3+n$.

$M_{g,n}^{trop}$ is clearly contractible (to the cone pt), here is something topologically more interesting

The link $\Delta_{g,n}$ of $M_{g,n}^{trop}$ is the subspace consisting of graphs of total edge length 1

Example:

$\Delta_2 \subset M_2^{trop}$

How to tropicalize $\mathcal{M}_{g,n}$?

$$\begin{tikzcd} \mathcal{M}_{g,n} \arrow[d] \\ \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

$\mathcal{M}_{g,n}$ moduli space of smooth curves of genus g / $\mathbb{C}$ with n markings

$\overline{\mathcal{M}}_{g,n}$ moduli space of stable curves ...

Problems → $\mathcal{M}_{g,n}$ does not embed into a toric variety in general

→ $\mathcal{M}_{g,n}$ is not a scheme "

Solution: use skeletons (Thuillier)

Consider $(K, |\cdot|_0)$, $K = \overline{K}$

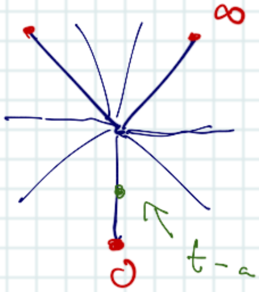
Let $X/_K$ and $D \subset X$ strict normal crossings.

Then we have

$$\begin{tikzcd} D^{an} \arrow[r, hook] \arrow[d, phantom, "\cup"] & X^{an} \\ \Delta(D) \end{tikzcd}$$

and a reduction map red: $X^{an} \longrightarrow X$

Examples ① $\left(\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0,1,\infty\}\right)$



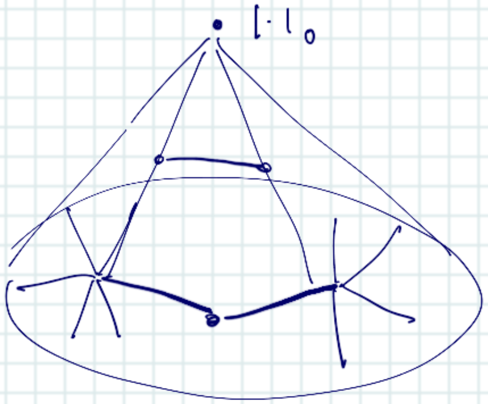
t-adic val on $K(t)$

lifts ! to $\mathrm{Spec}(\mathcal{H}(x)^\circ) \to X$

$\mathrm{Spec}(\widetilde{\mathcal{H}}(x)) \longrightarrow X$

② $\left(X \subset \mathbb{P}^2 \right)$

$(X)^{an} \subseteq \mathbb{P}^{2,an}$



What happens when we have self intersections?

X smooth $\qquad D \hookrightarrow X$ normal crossings

(étale locally $\cong \prod_{i=1}^{k} x_i = 0 \subset \mathbb{C}^n$)

e.g. $x^2y = z^2 \subset \mathbb{C}^3$ (Whitney umbrella)

- irreducible
- $y \neq 0 \rightsquigarrow$ locally given by $(z - \sqrt{y}\,x)(z + \sqrt{y}\,x) = 0$ /

$$D : x^2y = z^2 \subset \mathbb{C}^3 \setminus \{y = 0\} =: X$$

What should $\Delta(D)$ be?

Definition

Let $D \hookrightarrow X$ nc and $U \twoheadrightarrow X$ étale surjective s.th. $\tilde{D} := U \times_X D$ is snc in U

Let $D_2 := \tilde{D} \times_X \tilde{D}$ and $U_2 := U \times_X U$

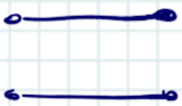
$$\Delta(D) := \mathrm{Coeq}_{\mathrm{Top}}\left(\Delta(D_2 \hookrightarrow U_2) \rightrightarrows \Delta(\tilde{D} \hookrightarrow U)\right)$$

Back to example Normalization: $y = u^2$

$U \longrightarrow X$ $\tilde{D} := U \times_X D$ $(z - ux)(z + ux) = 0$

$\Delta(\tilde{D})$:

$\Delta(D_2)$:



(and the two arrows are flipped)

$D_2 = \tilde{D} \times \mathbb{Z}/2\mathbb{Z}$ $\Rightarrow \Delta(D)$:

Theorem (Thuillier): Let (X, D) be toroidal $\left(\text{e.g. } nc\right)$

Let $X^D = \{ x \in X^{an} \,/\, red(x) \in D \}$

∃ a canonical deformation retraction of

$X^D \setminus D^{an}$ onto $\Delta(D) \times \mathbb{R}_{>0}$.

Application:

$(Y, D) \xrightarrow{\rho} (X, X^{sing})$ resolution $\left(\begin{matrix}\text{is iso} \\ Y \setminus D \simeq X \setminus X^{sing}\end{matrix}\right)$

⇒ homotopy type of $\Delta(D)$ does not depend on ρ

Fact $X \setminus D = \mathcal{M}_{g,n} \hookrightarrow \overline{\mathcal{M}}_{g,n}$ is an embedding with normal crossings

Abramovich-Caporaso-Payne: extend Thuillier's theorem to the case of toroidal DM stacks.

Theorem $\Delta_{g,n}$ can be identified with the boundary complex of $\overline{\mathcal{M}}_{g,n}$ and we have a comm. diagram

$$\begin{tikzcd} \overline{\mathcal{M}}^{an}_{g,n} \arrow[r] \arrow[dr] & M^{trop}_{g,n} \arrow[d, phantom, "\parallel"] \\ & \text{Cone over } \Delta_{g,n} \end{tikzcd}$$

To better understand why $\Delta_{g,n}$ is the boundary complex of $M_{g,n} \hookrightarrow \overline{M}_{g,n}$ it is useful to stratify $D \doteq \overline{M}_{g,n} \setminus M_{g,n}$

Let's do the $n=0$ version for simplicity:

$$D = \bigcup \mathcal{M}_{(G,g)}$$

$v \in V(G) \rightsquigarrow n_v$ valence

$$\widetilde{\mathcal{M}}_G = \prod_{v \in V(G)} \mathcal{M}_{g(v), n_v}$$

<u>Fact</u> $\mathcal{M}_{(G,g,m)} \cong [\widetilde{\mathcal{M}}_G / \mathrm{Aut}(G)]$ (stack quotient)

$\overset{\varphi}{p}$ then D locally analytic is $x_1 \cdot \ldots \cdot x_d = 0$

How many? $x_1 = 0$ corresponds to an edge of G & parametrizes local smoothings of the node.

$\rightsquigarrow$ here the bdry complex is $\Delta^{E(G)-1}$ (standard simplex)

plus monodromy coming from $\Delta(D \times D) \rightrightarrows \Delta(D)$ so in the end we get $\Delta^{E(G)-1}/\mathrm{Aut}(G)$.

Applications to the cohomology of $\mathcal{M}_g$

Theorem

There is a surjection

$$H^{4g-6}(\mathcal{M}_g;\mathbb{Q}) \longrightarrow \tilde{H}_{2g-1}(\Delta_g;\mathbb{Q})$$

Theorem (CGP '18)

$$H^{4g-6}(\mathcal{M}_g,\mathbb{Q}) \neq 0 \quad \text{for odd } g$$

Disproves conjectures by Church-Farb-Putman and Kontsevich!

Top weight cohomology .

$H^{k}(\mathcal{M}_{g}, \mathbb{Q})$ has a weight filtration $(W_{\bullet} H^{k}(-))$
(cf. Deligne's theory of mixed Hodge structures)

$$\mathrm{Gr}^{W}_{j} H^{k}(\mathcal{M}_{g}, \mathbb{Q}) := W_{j} H^{k}(\mathcal{M}_{g}, \mathbb{Q}) \Big/ W_{j-1}(-)$$

Theorem

$$\widetilde{H}_{i}(\Delta_{g}, \mathbb{Q}) \cong \mathrm{Gr}^{W}_{6g-6} H^{6g-6-(i+1)}(\mathcal{M}_{g}, \mathbb{Q}) .$$

(Deligne) ↘

Proof . this works for all boundary complexes of $D \hookrightarrow X$ smooth proper connected, $\dim(X)=n$

(homological algebra $\rightsquigarrow$ $W_0 H^{i}(D, \mathbb{Q}) \cong H_i(\Delta(D), \mathbb{Q})$)

but one also has

$$W_0 H^{i}(X, \mathbb{Q}) \to W_0 H^{i}(D, \mathbb{Q}) \longrightarrow W_0 H^{i+1}_{c}(X \setminus D, \mathbb{Q}) \to W_0 H^{i+1}(X,\mathbb{Q}$$

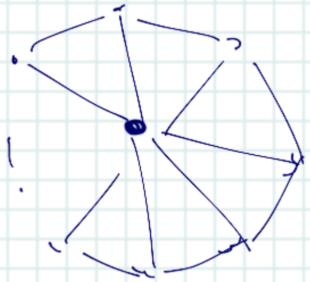
$i=0$ | $W_0 H^{i}(X, \mathbb{Q}) = \mathbb{Q}$, $W_0 H^{i+1}(X,\mathbb{Q}) = 0$ b/c smooth

$$\Rightarrow \quad W_0 H^{i+1}_{c}(X \setminus D, \mathbb{Q}) = \widetilde{H}_{i}(\Delta(D), \mathbb{Q})$$

Poincaré duality
$$\Rightarrow \quad \mathrm{Gr}^{W}_{2n} H^{2n-(i+1)}(X \setminus D, \mathbb{Q}) = \widetilde{H}_{i}(\Delta(D), \mathbb{Q})$$

Non-vanishing of $H^{4g-6}(\mathcal{M}_g, \mathbb{Q})$ (CGP)

"proof": let W_g be the g-wheel graph



Idea: show that $[W_g] \neq 0$ in $\widetilde{H}_{2g-1}(\Delta_g; \mathbb{Q})$.

[This was done by Willwacher for (W_g) in $\widetilde{H}_0(G^{(g)}, \mathbb{Q})$]

So, the key is to establish an isomorphism

$$\widetilde{H}_{2g-1}(\Delta_g; \mathbb{Q}) \simeq \widetilde{H}_0(G^{(g)} \mathbb{Q})$$

which can be done by showing that the subspace of Δ_g consisting of graphs with $g(v) > 0$ for some v has vanishing cohomology

(see CGP §4 for all the details).